# Diophantine Approximation on varieties V: Algebraic independence criteria

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## Contents



# 1 Introduction

Let $\mathbb{P}^M$ be the projective space of dimension $M$ over Spec $\mathbb{Z}$, and $\mathcal{X}$ an irreducible arithmetic sub variety. A point $\theta \in \mathcal{X}(\mathbb{C})$ is called generic, if the algebraic closure of $\{\theta\}$ over $\mathbb{Z}$ is all of $\mathcal{X}$. Part III of this series of papers ([Ma3]) established a lower bound for the approximability of generic points $\theta$ by algebraic points or sub varieties in terms of the dimension of $\mathcal{X}$, which by [Ma5] is best possible except for a subset of points of measure zero. More specifically, if the height and degree of an effective cycle on $\mathcal{X}$ are defined via $O(1)$ and $\overline{O(1)}$, and the algebraic distance of an effective cycle to $\theta$ is defined with respect to $\mu = c_1(\overline{O(1)})$. (See [Ma1], Section4)

**1.1 Theorem** *Let $\mathcal{X}$ be an irreducible quasi projective arithmetic variety of relative dimension $t$ over Spec $\mathcal{O}_k$, and $\bar{\mathcal{L}}$ an ample positive metrized line bundle on some projective compactification of $\mathcal{X}$. There is a number $b > 0$ such that for every $a >> 0$, and every generic $\theta \in X(\mathbb{C}_\sigma)$ there is an infinite subset $M \subset \mathbb{N}$ such that for each $D \in M$ there is an irreducible subscheme $\alpha_D$ of codimension $t$ fulfilling*

$$\deg \alpha_D \leq D^t, \quad h(\alpha_D) \leq aD^t, \quad \log|\alpha_D, \theta| \leq -baD^{t+1}.$$

Proof [Ma3], Theorem 1.2, Corollary 1.3.

It is the objective of this paper to reverse this conclusion, i. e. the approximability of a generic point by algebraic subvarieties will imply a lower bound on the dimension of $\mathcal{X}$, and hence give criteria of algebraic independence of complex numbers in terms of the approximability of corresponding points on arithmetic varieties.
For the rest of the paper, if not specified otherwise, $\partial_1, \ldots, \partial_t$ will be derivations of $k(X)$ whose restrictions to the tangent space of $X$ at $\theta$ are linearily independent. For a multi index $I = (i_1, \ldots, it) \in \mathbb{N}^t$, denote by $|I|$ its norm $i_1 + \cdots + i_t$ and by $\partial^I$ the differential operator $\partial_1^{i_1} \cdots \partial_t^{i_t}$. Further, for a global section $f \in \Gamma(X, O(D))$ denote $|f|_{L^2(\mathbb{P}^M)} = (\int_{\mathbb{P}^M} |f|^2 \mu^M)^{1/2}$.

**1.2 Theorem** *Let $\mathcal{X}$ be an irreducible subvariety of relative dimension $t$ in $\mathbb{P}^M$, and $\theta = [(\theta_0, \ldots, \theta_m)] \in X(\mathbb{C}_\sigma)$ a generic point. One may assume $\theta_0 \neq 0$, and then $t = trdeg_k(\theta_1/\theta_0, \ldots, \theta_M/\theta_0)$. Let further, $D_k, S_k$ be series of natural numbers, $H_k, V_k$ series of positive real numbers such that $S_k \leq D_k$, the series $D_k/S_k, H_k/S_k, V_k/S_k$ are non- decreasing, and*

$$\limsup_{k \to \infty} \frac{S_k^s V_k}{D_k^s (D_k + H_k)} = \infty.$$

*Additionally assume that for each sufficiently big $k \in \mathbb{N}$, there is a set of global sections $\mathcal{F}_k$ of $O(D)$ such for each $f \in \mathcal{F}_k$,*

$$\deg f \leq D_k, \quad \log|f|_{L^2(\mathbb{P}^>M)} \leq H_k, \quad sup_{|I| \leq S_k} \log|\partial^I(f/g^{\otimes D})(\theta)| \leq -V_k.$$

*and that there is no point $x \in \mathbb{P}^M(\mathbb{C})$ such that $f_x = 0$ for every $f \in \mathcal{F}_k$, and $\log|x, \theta| \leq \frac{V_{k-1}}{S_{k-1}}$. Then $t$ is at least $s + 1$.*

The criterion entails the Philippon criterion if one takes $S_k = 0$ for all $k$. An alternative proof to the one given here was already given in [LR] (Theorem 2.1). Under an additional assumption, this new proof furthermore also entails a characterisation of the point $\theta$ in terms of its approximability.
This criterion has a difficiency because it is usually used in cases in which the series $(D_k, H_k, S_k, V_k)$ fulfill certain regularity conditions (see below), and in this case there verifyably are points $\theta$ on any variety $\mathcal{X}$ that fulfill the conclusion of Theorem 1.2 without fulfilling its premiss.

**1.3 Definition** *A function $f : \mathbb{N} \to \mathbb{N}$ $(\mathbb{R})$ is said to be of uniform polynomial growth, if the limes*

$$n_f = \lim_{k\to\infty} \frac{k(f(k+1) - f(k))}{f(k)}$$

*exists.*

**1.4 Lemma**

1. *The set of functions of uniform polynomial growth is closed under compositions, sums, products, differences and quotients with*

   $$n_{f\circ g} = n_f n_g, \quad n_{f+g} = max(n_f, n_g), \quad n_{fg} = n_f + n_g, \quad n_{1/f} = -n_f, \quad n_{-f} = n_f.$$

   *If $f$ is unbounded, $f^{-1}$ is defined via*

   $$f^{-1}(n) = \inf\{k | f(k) \geq n\},$$

   *and $f$ is of uniform polynomial growth with $n_f \neq 0$, then $f^{-1}$ is of uniform polynomial growth with $n_{f^{-1}} = 1/n_f$.*

2. *If $f, g$ are of uniform polynomial growth, and $f(k) \geq g(k)$ for every sufficiently big $k$, then $n_f \geq n_g$.*

3. *A function $f$ is of uniform polynomial growth, if and only if there is an $n_f \in \mathbb{R}$ such that for every $\epsilon > 0$, there is a $k_0 \in \mathbb{N}$ such that*

   $$k^{n_f - \epsilon} < f(k) < k^{n_f + \epsilon}$$

   *for every $k \geq k_0$.*

4. *If $f$ is of uniform polynomial growth, and $n$ is any natural number, then for sufficiently big $k$,*

   $$f(k+n) \leq 2f(k).$$

**1.5 Definition** *Let $(D_k, S_k, H_k, V_k)$ be a quadrupel of sequences of natural and positive real numbers with $S_k \leq D_k/3$. The quadrupel is said to be of regular polynomial growth if $D_k/S_k$ and $H_k/S_k$ are monotonously increasing and unbounded, and the functions $f(k) = D_k/S_k$ and $g(k) = H_k/D_k$ are of uniform polynomial growth with $n_f > 0$, and $g(k) \geq c > 0$ for sufficiently big $k$.*

**1.6 Proposition** *In the situation of Theorem 1.2, if additionally* $(D_k, S_k, H_k, V_k)$ *is of regular polynomial growth, and* $trdeg_k(\theta) = s+1$*, then* $\theta$ *is an* $S$*-point in the sense of Mahler classification*

Proof [Ma5]

**1.7 Theorem** *Let* $\mathcal{X}$ *be a subvariety of relative dimension* $t$ *of* $\mathbb{P}^M$*, and* $\theta \in \mathcal{X}(\mathbb{C})$ *a generic point. Further,* $D_k, H_k, S_k, V_k$ *a quadrupel of sequences of natural and positive real numbers that is of regular polynomial growth, and fulfills*

$$\lim_{k\to\infty} \frac{S_k^s V_k}{D_k^s(D_k + H_k)} = \infty.$$

*Assume that for every sufficiently big* $k$*, there is a set* $\mathcal{F}_k \subset \Gamma(\mathbb{P}^M, O(D))$*, such that for every irreducible subvariety* $\mathcal{Y} \subset \mathcal{X}$ *that has sufficiently small distance to* $\theta$*, there is an* $f \in \mathcal{F}_k$*, and an* $I$ *with* $|I| \leq S_k/3$ *such that the restriction of* $\partial^I f$ *to* $\mathcal{Y}$ *is nonzero, and*

$$\log|f_k| \leq H_k, \quad \sup_{|I|\leq S_k} \log|\partial^I(f/g^{\otimes D}(\theta)| \leq -V_k.$$

*Then,* $t \geq s+1$*.*

**Remark:** The conditions in Theorem 1.7 are fulfilled e. g. if for every sufficiently big $k$ there are $t$ global sections $f_1, \ldots, f_t$ of $\mathcal{L}^{\otimes D_k}$ with

$$\log|f_i| \leq H_k, \quad D^{S_k}(\mathrm{div} f_i, \theta) \leq -V_k, \quad i = 1, \ldots, t,$$

and numbers $I_1, \ldots, I_t$ with $I_i \leq S_k$ such that the divisors of the sections $\partial^{I_1} f_1, \ldots, \partial^{I_t} f_t$ intersect properly. Another important case, where the conditions of the Theorem are fulfilled, will be when the global sections with small algebraic distance are obtained by having high order of vanishing at a certain point, and behave well with respect to differentiation.

# 2 Prerequisites

**2.1 Lemma** *Let* $\mathcal{X}$ *be a regular projective arithmetic variety,* $\bar{\mathcal{L}}$ *a metrized line bundle on* $\mathcal{X}$*, and* $f$ *a global section of* $\mathcal{L}^{\otimes D}$*. Then, for every effective cycle* $\mathcal{Z}$ *on* $\mathcal{X}$ *such that the intersection of* $\mathcal{Z}$ *with divf is proper,*

$$h(divf.\mathcal{Z}) = Dh(\mathcal{X}) + \int_Z \log|f| c_1(\bar{L})^m,$$

*where $m$ is the dimension of $Z$. In particular, if $\mathcal{Z}$ is an effective cycle of pure codimension in projective space, and $f \in \Gamma(\mathbb{P}^t, O(D))$, then*

$$h(divf) = Dh(\mathcal{Z}) + \int_Z \log|f|\mu^m,$$

*with $\mu = c_1(\bar{L})$.*

Proof [BGS], Proposition 3.2.1.(iv).

**2.2 Lemma** *Let $\mathcal{X}, \mathcal{Y}$ be regular projective arithmetic varieties, and $f : \mathcal{X} \to \mathcal{Y}$ a morphism. Then for every metrized line bundle $\bar{\mathcal{L}}$ on $\mathcal{Y}$, and every cycle $\mathcal{Z}$ in $\mathcal{X}$, such that* $\dim f(\mathcal{Z}) = \dim \mathcal{Z}$,

$$h_{f^*\bar{\mathcal{L}}}(\mathcal{Z}) = h_{\mathcal{L}}(f_*\mathcal{Z}).$$

Proof [BGS], Proposition 3.2.1.(iii).

**2.3 Lemma** *For $m < n$ let $\mathbb{P}^m \subset \mathbb{P}^n$ the projective subspace corresponding to a choice of $m+1$ homogeneous coordinates, and $\mathbb{P}^{n-m-1} \subset \mathbb{P}^n$ the subspace corresponding the remaining $n-m$ coordinates. With $\pi$ the map*

$$\pi : \mathbb{P}^n \setminus \mathbb{P}^m \to \mathbb{P}^{n-m-1}, \quad [v+w] \mapsto [v], \quad [v] \in \mathbb{P}^m, [w] \in \mathbb{P}^{n-m-1},$$

*and any cycle $\mathcal{Z}$ in $\mathbb{P}^n$, such that $Z$ does not meet $\mathbb{P}^m$,*

$$h(\pi_*(Z)) \leq h(Z).$$

Proof [BGS], (3.3.7).

**2.4 Lemma** *For every $f \in \Gamma(\mathbb{P}^t_{\mathbb{C}}, O(D))$,*

$$\log|f|_\infty - \frac{D}{2}\sum_{m=1}^{t}\frac{1}{m} \leq \int_{\mathbb{P}^t_{\mathbb{C}}} \log|f|\mu^t \leq \log|f|_{L^2} \leq \log|f|_\infty.$$

Proof [BGS], (1.4.10).

**2.5 Lemma** *Let $f \in \Gamma(\mathbb{P}^t, O(D)), g \in \Gamma(\mathbb{P}^t, O(D'))$. Then,*

$$\log|f|_{L^2} + \log|g|_{L^2} - c_2(\log D + \log D') \leq \log|fg|_{L^2} \leq$$

$$\log|f|_{L^2} + \log|g|_{L^2} + c_1(D+D') + \log\binom{D+D'+t}{t}.$$

Proof [Ma2], Lemma 3.2.

For $p \leq t$, $Z \in Z^p_{eff}(\mathbb{P}^t)$, and $\theta$ a point not contained in the support of $Z$ in [Ma1] the algebraic distance $D(Z,\theta)$, is defined. Recall also the definition of the derivated algebraic distance of a point $\theta$ to an effective $X$ cycle in $\mathbb{P}^t$, whose support does not contain $\theta$ in [Ma4]: Let $I = (1_1, \ldots, i_{2t}) \in \mathbb{N}^{2t}$ denote a multi index, $|I| = i_1 + \cdots + i_{2n}$ its norm, and $\partial^I$ the differential operator $\partial^{i_1}/\partial x_1 \partial^{i_2}/\partial y_i \cdots \partial^{i_{2t}}/\partial y_t$, and let $\varphi : \mathbb{A}^t(\mathbb{C}) \to \mathbb{P}^t(\mathbb{C})$ be the affine chart with $\varphi(0) = \theta$. The derivated algebraic distance $D^S(Z,\theta)$ of order $S \in \mathbb{N}$ is defined as

$$D^S(\theta, X) := sup_{|I| \leq S} \log |\partial^I \exp D(\theta, X)|.$$

If $\psi$ is another affine chart centered at $\theta$, the derivated algebraic distance with respect to $\psi$ differs from that with respect to $\varphi$ only by a constant depending on $\psi$ and $\varphi$ times $S \log \deg X$. See [Ma4].
There are the following Propositions for the derivated algebraic distance.

**2.6 Proposition** *For $s, D \in \mathbb{N}$, and $f \in \Gamma(\mathbb{P}^t, O(D))$ let $F$ be the polynomial of degree at most $D$ in $t$ variables that corresponds to $f$ with respect to affine coordinates of $\mathbb{P}^t$ centered at $\theta$. Then, with some positive constant $c$ only depending on $t$,*

$$D^S(divf, \theta) = \sup_{s \leq S |J|=s} \log \left| \left( \frac{\partial^s}{(\partial z_1)^{j_1} \cdots (\partial z_t)^{j_t}} F \right)(0) \right| - \log |f| + O((S+D)\log(SD)).$$

Proof [Ma4], Theorem 1.3.

**2.7 Corollary**

In the situation of the Lemma,

$$D^S(\mathrm{div} f, \theta) \leq \sup_{|J| \leq S} \log \left| (\partial^J F)(0) \right| + c(S+D)\log(SD).$$

Proof Follows from the estimate

$$\log |f| \geq -cD$$

for global sections $f$ of $O(D)$ with a fixed positive constant $c$.
We will need two special cases of the derivative metric Bézout Theorem, proved in [Ma4], namely

**2.8 Theorem** *Let $\mathcal{X}, \mathcal{Y}$ be properly intersecting effective cycles in projective space $\mathbb{P}^t_{\mathbb{Z}}$, and $S, \bar{S}$ natural numbers with $S \leq \deg X/3, \bar{S} \leq \deg Y/3$. There is a positive constant $d$ only depending on $t$, and a function $f$ from the set of natural numbers less or equal $\deg X + \deg Y$ to the set of pairs of natural nubers less or equal $\deg X$ and $\deg Y$ respectively, such that $pr_1 \circ f$ and $pr_2 \circ f$ are surjective, and for every $\theta \in \mathbb{P}^t(\mathbb{C})$ not contained in the support of $X.Y$.*

1. *For a given $k_0 \leq \deg Z_0 + \deg Z_1$, and any $k \leq \deg Z_0 + \deg Z_1$ greater or equal $k_0$, and $(\bar{\nu}_0, \bar{\nu}_1) = f(k)$,*

$$2(\bar{\nu}_0 - \nu_0)(\bar{\nu}_1 - \nu_1) \log |Z_0 + Z_1, \theta| + 2D^S(Z_0.Z_1, \theta) + 2D(Z_0, Z_1) \leq$$

$$(\bar{\nu}_0 - \nu_0) D^{3\nu_1}(Z_1, \theta) + (\bar{\nu}_1 - \nu_1) D^{3\nu_0}(Z_0, \theta) +$$

$$O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)).$$

2. 

$$2D(X, Y) + 2D^{S\bar{S}}(X.Y, \theta) \leq$$

$$max(\bar{S} D^{3S}(X, \theta), S D^{3\bar{S}}(Y, \theta)) + d(\deg X \deg Y) \log(\deg X \deg Y),$$

*and*

$$2D(X, Y) + 2D(X.Y, \theta) \leq$$

$$max(\bar{S} D(X, \theta), D^{3\bar{S}}(Y, \theta)) + d(\deg X \deg Y) \log(\deg X \deg Y).$$

Proof [Ma4], Theorem 1.9, Corollary 1.11.

**2.9 Corollary**

1. *For $S_0, d_0 \leq \deg Z_0/3, S_1 \leq Z_1/3$, and $S = S_0 S_1$, there is a $K \leq d_0 S_1$ such that*

$$K \log |Z_0 + Z_1, \theta| + 2D^S(Z_0.Z_1, \theta) + 2D(Z_0, Z_1) \leq$$

$$max(S_1 D^{9S_0}(Z_0, \theta), d_0 D^{9S_1}(Z_1, \theta) +$$

$$O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)).$$

2. *For $S_0 \leq \deg Z_0/3$, $S_1 \leq \deg Z_1/3$, and $|Z_0, \theta| \leq |Z_1, \theta|$,*

$$2D^{S_1}(Z_0, Z_1) + 2D(Z_0, Z_1) \leq$$

$$D^{3S_1}(Z_1, \theta) + O((\deg Z_0 \deg Z_1 + S) \log(S \deg Z_0 \deg Z_1)).$$

Proof The proof is similar to the one of [Ma4], Corollary 1.11.

Similar to [Ma3], Proposition 2.4.1, on can also deduce

**2.10 Theorem** *Let $\mathcal{Y}$ be an irreducible effective cycle of codimension $p$ in projective space, $f \in \Gamma(\mathbb{P}^t, O(D))_{\mathbb{Z}}$ a global section whose restriction to $\mathcal{Y}$ is nonzero, and $\bar{f} \in \Gamma(\mathbb{P}^t, O(D))_{\mathbb{R}}$ a global section that is orthogonal to $I_{\mathcal{Y}}(D)$ the elements of degree $D$ in the vanishing ideal of $Y$ such that $f_Y = \bar{f}_Y$. Then for natural numbers $S \leq \deg Y/3, \bar{S} \leq D/3$ such that for every $\theta \in \mathbb{P}^t(\mathbb{C})$ not contained in $divf.Y$,*

$$2D^{S\bar{S}}(Y.divf, \theta) \leq$$

$$max(\bar{S}D^S(div\bar{f}, \theta), SD^{\bar{S}}(Y, \theta)) + \deg Y \log |f_Y^\perp| + Dh(\mathcal{Y}) + dD \deg Y \log(D \deg Y),$$

*and*

$$D(Y.divf, \theta) \leq max(\bar{S}D(div\bar{f}, \theta), D^{\bar{S}}(Y, \theta)) + dD \deg Y \log(dD \log Y).$$

Let $H : \mathbb{N} \to \mathbb{R}$ be function of uniform polynomial growth such that $H(D)/D \geq a > 0$ with a sufficiently big constant $a$, hence by Lemma 1.4.4, $n_H \geq 0$.
For $\mathcal{X}$ an effective cycle in $\mathbb{P}^t$ define the $H/D$-size of $\mathcal{X}$ as

$$t_{\frac{H}{D}}(\mathcal{X}) = \frac{H}{D} \deg X + h(\mathcal{X}).$$

**2.11 Proposition** *There are constants $c, b, \bar{b} > 0, n \in \mathbb{N}$ only depending on $t$ such that for every generic $\theta \in \mathbb{P}^t(\mathbb{C})$ and every function $H : \mathbb{N} \to \mathbb{R}$ as above, there is an infinite set $M \subset \mathbb{N}$ such that for every $D \in \mathbb{N}$, there is an irreducible zero dimensional subvariety $\alpha_{nD}$ of $\mathbb{P}^t_{\mathbb{Z}}$, a locally complete intersection $\mathcal{X}$ of codimension $s \leq t-1$ at $\alpha_D$ and global sections $f \in \Gamma(\mathbb{P}^t, O(D))_{\mathbb{Z}}, \bar{f} \in \Gamma(\mathbb{P}^t, O(D))_{\mathbb{R}}$ such that $f^\perp_{\alpha_{nD}} = \bar{f}^\perp_{\alpha_{nD}} \neq 0$, and with $\mathcal{X}_{min}$ the irreducible component of $\mathcal{X}$ with minimal $H/D$-size,*

$$\deg \mathcal{X} \leq D^s, \quad h(\mathcal{X}) \leq HD^{s-1},$$

$$\log |\bar{f}^\perp_{\alpha_{nD}}| \leq H, \quad \log |\langle f|\theta \rangle| \leq -bt_{\frac{H}{D}}(\mathcal{X}_{min})D^{t+1-s},$$

$$\deg \alpha_{nD} \leq (nD)^t, \quad h(\alpha_{nD}) \leq (nH)(nD)^{t-1}, \quad D(\alpha_{nD}, \theta) \leq -\bar{b}t_{\frac{H}{D}}(\alpha_{nD})D,$$

$$\log |\alpha_{nD}, \theta| \leq -\bar{b}t_{\frac{H}{D}}(\alpha_{nD})D, \quad t_{\frac{H}{D}}(\alpha_{nD}) \geq ct_{\frac{H}{D}}(\mathcal{X}_{min})D^{t-s}.$$

Proof [Ma3], Corollary 4.21. One has to be cautious to adjust the constants.

Another important tool for the proofs is the Liouville inequality.

**2.12 Proposition: Liouville inequality** *Let $f \in \Gamma(\mathbb{P}^t, O(D))$, and $\alpha$ an algebraic point such that $f_\alpha \neq 0$. There is a constant $d$, only depending on $t$ such that*

$$D(divf, \alpha) \geq Dh(\alpha) - \deg \alpha \log |f| - dD \deg \alpha.$$

For the relation of this Proposition to the classical formulation of the Liouville inequality, compare [Ma6].

Proof Since by [Ma1], Theorem 2.2.2, $h(\text{div}f) \leq \log|f| + D\sigma_t$, this is a special case of the equality

$$D(\text{div}f, \alpha) = h(\text{div}f.\alpha) - \deg\alpha h(\text{div}f) - \deg f h(\alpha) + \sigma_t \deg f \deg\alpha$$

from [Ma1], Scholie 4.3, together with the estimate $D(\text{div}f, \alpha) \leq d' \deg\alpha \deg f$ from [BGS], Proposition 5.1.

# 3 Derivatives

## 3.1 Polynomials modelling derivatives of rational functions

With $\mathcal{X}$ an arithmetic sub variety of relative dimension $t$ in $\mathbb{P}^M_{\mathbb{Z}}$, and $g$ a global section of $O(1)$ whose restriction to $\mathcal{X}$ is nonzero, let $\theta \in X(\mathbb{C}_\sigma)$ be a generic point, and $\partial_1, \ldots, partial_t$ derivatives of $X$ as in the introduction.

**3.1 Lemma** *With the above notations, and $f$ a global section of $\mathcal{L}^{\otimes D}$,*

$$\sup_{|I|\leq S} \log\left|\partial^I \frac{f}{g^{\otimes D}}(\theta)\right| = D^S(divf, \theta) + \log|f|_{L^2(\mathbb{P}^M)} + O((S+D)\log SD),$$

*for every $S \leq D$.*

Proof Let $U_\theta$ be a neighbourhood of $\theta$, and $\varphi : U \to U_\theta$ an affine chart of $U_\theta$. Further, $\tilde{\partial}_1, \ldots \tilde{\partial}_t$ the canonical derivatives on $U$. Then,

$$(\varphi^{-1})^*\partial = h\tilde{\partial}$$

with a $(t \times t)$-matrix of rational functions $h$. Hence,

$$\sup_{|I|\leq S} \log\left|\partial^I \frac{f}{g^{\otimes D}}(\theta)\right| = \sup_{|I|\leq S} \log\left|\tilde{\partial}^I(\varphi^* f)(0)\right| + O((S+D)\log SD),$$

and the Lemma follows from [Ma4], Theorem 1.3.

**3.2 Corollary** *If $\bar{g}$ is another global section of $\mathcal{L}$, and $\bar{\partial}_1, \ldots, \bar{\partial}_t$ another set of derivations of $k(X)$ whose restrictions to $T_\theta X$ are linearily independent, then*

$$\sup_{|I|\leq S} \log\left|\partial^I \frac{f}{g^{\otimes D}}(\theta)\right| = \sup_{|I|\leq S} \log\left|\bar{\partial}^I \frac{f}{\bar{g}^{\otimes D}}(\theta)\right| + O((S+D)\log SD).$$

Because of this Corollary to the derivatives of a global section it doesn't matter which derivatives $\partial_1, \dots, \partial_t$ in $k(X)$ one choses. In the proofs of the main Theorem we will chose them according to the definition in the next Theorem.
Let $\mathcal{X} \subset \mathbb{P}^M_{\mathbb{Z}}$ be an irreducible subvariety of relative dimension $t$, and $\mathbb{P}^t \subset \mathbb{P}^M$ a subspace defined over Spec$\mathbb{Z}$ such $(\mathbb{P}^t)^\perp$ does not meet $\mathcal{X}$. Then, with

$$\pi : \mathbb{P}^M \setminus (\mathbb{P}^t)^\perp \to \mathbb{P}^t,$$

the restriction $\pi_X$ of $\pi$ to $\mathcal{X}$ is a proper map from $\mathcal{X}$ to $\mathbb{P}^t$. Denote by $x_0, \dots, x_M$ the homogeneous coordinates of $\mathbb{P}^M$ ordered such that $x_0, \dots, x_t$ are homogeneous coordinates of $\mathbb{P}^t$. There is the canonical map $k(x_1, \dots, x_M) \cong k(\mathbb{P}^M) \to k(X)$, and if $\bar{x}_i, i = 1, \dots, t$ denotes the image of $x_i$ under this map, the function field $k(X)$ is a finite extension of $k(\bar{x}_1, \dots, \bar{x}_t)$.
We denote by $\partial/\partial x_\mu$ the usual derivations of $k(x_1, \dots, x_M) \cong k(x_1/x_0, \dots, x_M/x_0)$, and do not distinguish between a polynomial $f(x_1, \dots, x_M)$, its image $f(x_1/x_0, \dots, x_M/x_0)$ in $k(\mathbb{P}^M)$, and its image $f(\bar{x}_1, \dots, \bar{x}_M)$ in $k(X)$. The following Theorem is a generalization of [RW], Proposition ?? to higher dimensions.

**3.3 Theorem** *With the above notations, let $\partial_t, \dots, \partial_t$ be the derivations of $k(X)$ defined by*

$$\partial_l x_l = 1, \quad \textit{and} \quad \partial_l x_i = 0 \quad \textit{for} \quad i \in \{1, \dots, t\} \setminus \{l\}.$$

*Let further $I = (i_1, \dots, i_t) \in \mathbb{N}^t$ be a multi index of degree $S = i_1 + \dots + i_t$, and $\partial^I = \partial_1^{i_1} \cdots \partial_t^{i_t}$.*
*There is a homogeneous polynomial $P = P(x_0, \dots x_M)$ with*

$$\deg P \leq (M - t) \deg X, \quad \log |P|_{L^2(\mathbb{P}^M)} \leq c \deg X + h(\mathcal{X}),$$

*with $c$ a constant only depending on $M$ and $t$, such that for every multi index $I$ of degree $S$, and every homogeneous polynomial $f$,*

$$\partial^I f = \frac{f_I}{P^{2S-1}}.$$

*where $f_I$ is a homogeneous polynomial with*

$$\deg f_I \leq \deg f + (2S - 1)(M - t) \deg X,$$

$$\log |f_I|_{L^2(\mathbb{P}^M)} \leq$$
$$\log |f| + \log \deg f + (2S - 1)(M - t)(h(\mathcal{X}) + c \deg X + \log \deg X) + \log(2S!).$$

Proof Let $\pi_X$ be the restriction of $\pi$ to $\mathcal{X}$. For any $\mu = t+1, \dots, M$, the projection of $\mathcal{X}$ to the space with homogeneous coordinates $x_0, \dots, x_t, x_\mu$ is a hyper surface of

degree at most $\deg X$. Let $P_\mu$ be the corresponding homogeneous polynomial in $x_0, \ldots, x_t, x_\mu$. Then $\deg P_\mu \le \deg X$, and by Lemma 2.1, 2.3 and 2.4,

$$\log |P_\mu| \le \int_{\mathbb{P}^M} \mu^M + c \deg X \le h(\pi_* \mathcal{X}) + c \deg X \le h(\mathcal{X}) + c \deg X. \tag{1}$$

Let further

$$A_0 := \prod_{\mu=t+1}^{M} \frac{\partial P_\mu}{\partial x_\mu},$$

and

$$A_{l\mu} := -\frac{\partial P_\mu}{\partial x_l} \left( \frac{\partial P_\mu}{\partial x_\mu} \right)^{-1} \in k(X), \tag{2}$$

for $l = 1, \ldots, t$, and $\mu = t+1, \ldots, M$. We have

$$\deg A_0 \le (M-t)(\deg X - 1),$$

and using Lemma 2.5, and (1),

$$\log |A_0| \le (M-t)(h(\mathcal{X}) + c \deg X + \log \deg X).$$

Also, $A_0 A_{l\mu}$ is a polynomial with

$$\deg(A_0 A_{k\mu}) \le (M-t)(\deg X - 1),$$

$$\log |A_0 A_{k\mu}| \le (M-t)(h(\mathcal{X}) + c \deg X + \log \deg X).$$

Since $P_\mu(x_1, \ldots, x_t, x_\mu) = 0$ on $X$, we get

$$0 = \partial_l P_\mu = \frac{\partial P_\mu}{\partial x_l} + \frac{\partial P_\mu}{\partial x_\mu} \partial_l x_\mu,$$

hence,

$$\partial_l x_\mu = -\frac{\partial P_\mu}{\partial x_l} \left( \frac{\partial P_\mu}{\partial x_\mu} \right)^{-1} = A_{l\mu},$$

and

$$f_l = A_0 \frac{\partial f}{\partial x_l} + \sum_{\mu=t+1}^{M} A_0 A_{l\mu} \frac{\partial f}{\partial x_\mu}$$

is a polynomial with

$$\deg f_l \le \max(\deg A_0, \deg(A_0 A_{l\mu})) + \deg f - 1 \le (M-t) \deg X + \deg f,$$

$$\log |f_l| \le \log \deg f + \log |f| + (M-t)(h(\mathcal{X}) + c \deg X + \log \deg X) + \log 2, \tag{3}$$

and

$$\partial_l f = \frac{f_l}{A_0}. \tag{4}$$

Put $P = A_0$. Then $\deg P \leq (M-t)(\deg X - 1)$, and the estimate on the norm of $P$ immediately follows from (1), and Lemma 2.5.
Assume now the Theorem proved for $I$ of degree $S$. That is, for any $I$ with $|I| = S$,

$$\partial^I f = \frac{f_I}{P^S},$$

for some polynomial $f_I$ with norm and degree fulfilling the estimates from the Theorem. Then, with $\bar{I} = I + (0, \ldots, 0, 1, 0, \ldots, 0)$,

$$\partial^{\bar{I}} f = \partial_l \partial^I f = \partial_l \frac{f_I}{P^{2S-1}} = \frac{(\partial_l f_I) P^{2S-1} - f_I (2S-1) P^{2S-2} \partial_l P}{P^{4S-2}} =$$

$$\frac{P^2 \partial_l f_I - (2S-1) f_I P \partial_l P}{P^{2S+1}}.$$

By (3), (4) and induction hypothesis $P^2 \partial_l f_I$, and $(S-1) P f_I \partial_l P$ are polynomials with

$$\begin{aligned}
\deg(P^2 \partial_l f_I) \leq \deg P + \deg f_I &\leq 2(M-t)\deg X + (2S-1)(M-t)\deg X \\
&+ \deg f \\
&= (2S+1)(M-t)\deg X + \deg f,
\end{aligned}$$

and

$$\begin{aligned}
\deg(f_I P \partial_l P) &\leq 2(M-t)\deg X + (2S-1)(M-t)\deg X + \deg f \\
&= (2S+1)(M-t)\deg X + \deg f.
\end{aligned}$$

Likewise, the norms of $P^2 \partial_l F_I$, and $(S-1) P f_I \partial_l P$ by (3); (4) and induction hypothesis fulfill the inequalities

$$\begin{aligned}
\log |P^2 \partial_l f_I| &\leq (2S+1)(M-t)(h(\mathcal{X}) + c \deg X + \log \deg X) + \log(2S)! \\
&+ \log |f| + \log \deg f \\
\\
&\leq 2(S+1)(M-t)(h(\mathcal{X}) + c \deg X + \log \deg X) + 2S \log 2S^2 \\
&+ \log |f| + \log \deg f,
\end{aligned}$$

and

$$\begin{aligned}
\log |(2S-1) f_I P \partial_k P| &\leq \log(2S-1) + (2S-1)(M-t) \times \\
&\quad (h(\mathcal{X}) + c \deg X + \log \deg X) \\
&+ \log |f| + \log \deg f + \log(2S)!.
\end{aligned}$$

Hence, with $f_{\bar{I}} = P^2\partial_l f_I - (2S-1)f_I P\partial_l P$, we have $\partial^{\bar{I}} f = f_{\bar{I}}/P^{2S+1}$, and

$$\deg f_{\bar{I}} \leq (S+1)(M-t)\deg X + \deg f,$$

$$\log|f_{\bar{I}}| \leq (2S+1)(M-t)(h(\mathcal{X}) + c\deg X + \log\deg X) + \log(2S+2)!,$$

and the claim follows for $S+1$.

**3.4 Corollary** *With the notations of the Theorem, for every $\theta$ in $\mathcal{X}(\mathbb{C})$, such that $f(\theta) \neq 0$ and $P(\theta) \neq 0$), there is a constant $c$ only depending on $\theta$, and $\mathcal{X}$ such that*

$$D^S(divf, \theta) = \sup_{|I|\leq S} \log|f_I(\theta)| + O((S+D)\log(SD)).$$

*Moreover, for every $T \leq S$,*

$$D^S(divf, \theta) = \sup_{|I|\leq S-T} \sup_{|J|\leq T} \log|(\partial^J f_I)(\theta)|.$$

Proof Since $\log|P(\theta)^{2S-1}| = c(2S-1)$, for some constant $c$, with $g = x_0^D$, the claim follows from the Theorem, together with Lemma 3.1.

## 3.2 Local Bézout Theorem

In this subsection $k$ is a field of characteristic zero and $X$ a scheme of dimension $t$ over Spec $k$. For $y$ a point in $X$ denote by $Y = \overline{\{y\}}$ its Zariski closure.

**3.5 Definition**

1. *Let $y$ be a point in $X$ with with $\dim Y = t-p$ . For $Z$ an irreducible subvariety of codimension $p-1$, $f \in k(Z)$ and $\mathfrak{m}_y \subset \mathcal{O}_{X,y}$ the maximal ideal in the localization of $\mathcal{O}_X$ at $y$, define the order of vanishing $v_y(f)$ of $f$ at $\mathcal{Y}$ as*

$$v_y(f) := max\{n \in \mathbb{N} | f \in \mathfrak{m}_y^n\}.$$

2. *For $X$ an irreducible subscheme of $\mathbb{P}^M_k$, and $\mathbb{P}(W) \subset \mathbb{P}^t = Projk[x_0, \ldots, x_M]$ a projective subspace of codimension $q$, let $w$ be the corresponding point in $\mathbb{P}^M$ and $Y$ an irreducible subvariety of codimension $p$ with $p \leq q$, define $v_w(Y)$ as*

$$v_w(Y) := min_{\mathbb{P}(F)}\{multiplicity\ of\ \mathbb{P}(W)\ in\ \mathbb{P}(F).Y\},$$

*where $\mathbb{P}(F)$ runs over all subspaces $\mathbb{P}(F) \subset \mathbb{P}^M$ of codimension $q-p$ that intersect $Y$ properly and contain $\mathbb{P}(W)$. Define $v_w : Z(\mathbb{P}^t) \to \mathbb{Z}$ by linear extension.*

3. *For $X = \mathbb{P}^M$, $w \in \mathbb{P}^M$ a point corresponding to a subspace $\mathbb{P}(W) \subset \mathbb{P}^M$, and $ZZ_1 - Z_2$ a cycle of pure codimension $p$ in $\mathbb{P}^M$ define the order of vanishing of $Z$ at $w$ as the difference of the orders of vanishing as defined in part 1 of the chow forms $f_{Z_1}, f_{Z_2}$ of $Z_1, Z_2$ at the subvariety*

$$\mathbb{P}(\check{W})_i = \check{\mathbb{P}}^M \times \cdots \times \check{\mathbb{P}}^M \times \mathbb{P}(\check{W}) \times \check{\mathbb{P}}^M \times \cdots \times \check{\mathbb{P}}^M,$$

*where $\check{\mathbb{P}}^M$ is the space dual to $\mathbb{P}^M$, and $\mathbb{P}(\check{W})$ the space dual to $\mathbb{P}(W)$. Since the chow divisor is invariant under permutation of the factors in $(\check{\mathbb{P}}^t)^{M+1-p}$, this number does not depend on the choice of $i \in \{1, \dots, M+1-p\}$.*

### 3.6 Lemma

1. *For $y_w$ the point corresponding to a subspace $\mathbb{P}(W) \subset \mathbb{P}^M$ of codimension $q$, and $Z$ a subvariety of codimension $q - 11$ in $\mathbb{P}^t$ the definitions in 1 and 2 coincide.*

2. *The Definitions 2 and 3 coincide.*

### 3.7 Fact

1. *If $w$ is a point corresponding to a subspace, $X$ an effective cycle in $\mathbb{P}^M$, then $\mathbb{P}(W) \subset$ supp $X$, if and only if $v_w(X) \geq 1$.*

2. *If $y$ is a closed point of $\mathbb{P}^M$, and $X$ an effective cycle of pure codimension $M$, the multiplicity of $y$ in $X$ equals $v_y(X)$.*

3. *Let $q \geq p$, and $\mathbb{P}(W), \mathbb{P}(F)$ be subspaces of codimension $q$, and $p$ respectively. If $w$ is the point corresponding to $\mathbb{P}(W)$, then*

$$v_w(\mathbb{P}(F)) = 1 \Leftrightarrow \mathbb{P}(W) \subset \mathbb{P}(F), \quad \text{and} \quad v_w(\mathbb{P}(F)) = 0 \Leftrightarrow \mathbb{P}(W) \not\subset \mathbb{P}(F).$$

4. *Let $\mathbb{P}(W) \subset \mathbb{P}(F) \subset \mathbb{P}^M$ be subspaces, and $Y$ an effective cycle intersecting $\mathbb{P}(F)$ properly. If $v_w^{\mathbb{P}(F)}(Y)$ is defined as the order of vanishing of $\mathbb{P}(F).Y$ at $\mathbb{P}(W)$ inside $\mathbb{P}(F)$, then*

$$v_w^{\mathbb{P}(F)}(Y) \geq v_w(Y).$$

5. *Let $X$ be an effective cycle of pure codimension $p$ in $\mathbb{P}^t$, and $y$ a closed point. Then for every subspace $\mathbb{P}(F) \subset \mathbb{P}^t$, of codimension $t - p$ containing $y$, and intersecting $Y$ properly,*

$$v_y(X) \leq v_y(\mathbb{P}(F).X),$$

*and there exists a subspace $\mathbb{P}(F)$ with these properties such that*

$$v_y(X) = v_y(\mathbb{P}(F).X).$$

**3.8 Proposition** *Let $X$ be an irreducible subvariety of dimension $t$ of $\mathbb{P}^M$, and $w$ a closed point in $X$. Further, $f, g \in \Gamma(\mathbb{P}^M, O(D))$ with $f_y \neq 0$. If for a natural number $S$ and every multi index $I$ with $|I| \leq S$ the equality $(\partial^I f)(y) = 0$ holds, then the order of vanishing $v_y(Z)$ of $Z = X.divf$ at $y$ is at least $S$.*

Proof By Fact 3.7, there is a subspace $\mathbb{P}(F) \subset \mathbb{P}^M$ be of codimension $t-1$ containing $y$ and properly intersecting $Z$ such that $v_y(Z) = v_y(\mathbb{P}(F).Z)$. Since $g_y \neq 0$, the multiplicity of $y$ in $\mathbb{P}(F).X.\mathrm{div}f$ equals the multiplicity of $y$ in $\mathbb{P}(F).X.\mathrm{div}(f/g)$, that is

$$v_y(Z) = v_y(\mathbb{P}(F).Z) = v_y(\mathbb{P}(F).X.\mathrm{div}(f/g)).$$

Further, if $\bar{f}, \bar{g}$ are the restrictions of $f, g$ to one-dimensional subvariety $\mathbb{P}(F) \cap X$, then

$$v_y(\mathbb{P}(F).X.\mathrm{div}(f/g)) \geq v_y(\mathrm{div}(\bar{f}/\bar{g})).$$

If $\partial$ is a derivation of $\mathbb{P}(F) \cap X$ whose restriction to $y \in \mathbb{P}(F) \cap X$ is nonzero, then $\partial$ is a linear combination with coefficients in $k(X)$ of $\partial_1, \ldots, \partial_t$, hence $\partial^s(\bar{f}/\bar{g}) = 0$ for every $s \leq S$, which means that $(\bar{f}/\bar{g})$ is contained in the $S$th power $\mathfrak{m}^S_{\mathbb{P}(F)\cap X,y}$ of the maximal ideal $\mathfrak{m}_{\mathbb{P}(F)\cap X,y} \subset \mathcal{O}_{\mathbb{P}(F)\cap X,y}$, that is $v_y(\bar{f}/\bar{g}) \geq S$. Together with the above equalities and estimates this implies the claim.

Two effective cycles $Y, Z$ of projective space are said to intersect properly at a point $x \in \mathbb{P}^M$ if for every irreducible component $U$ of the intersection of the supports of $Y$ and $Z$ that contains $x$ the equality $\mathrm{codim} W = \mathrm{codim} Y + \mathrm{codim} Z$ holds.

**3.9 Local Bézout Theorem** *For $x$ a closed point in $\mathbb{P}^M$ and two cycles $Y, Z$ of $\mathbb{P}^M$, intersecting properly at $x$,*

$$v_x(Y.Z) \geq v_x(Y)v_x(Z).$$

**3.10 Remark** *By Fact 3.7, the Theorem holds in case $Y$ is a projective subspace $\mathbb{P}(F) \subset \mathbb{P}^t$.*

**3.11 Lemma** *Let $Y, Z$ be properly intersecting irreducible varieties of codimension $p, q$ of $\mathbb{P}^M$, and $X\#Y$ their join. For a closed point $x$ in $\mathbb{P}^m$ there are subpaces $\mathbb{P}(F), \mathbb{P}(F')$ of codimensions $t-p, t-q$ containing $x$ such that the intersections $\mathbb{P}(F).Y$ and $\mathbb{P}(F').Z$ are proper, and*

$$v_{(x,x)}(Y\#Z) = v_{(x,x)}(Y\#Z.\mathbb{P}(F)\#\mathbb{P}(F')).$$

**3.12 Lemma** *A point $y\#z$ in $\mathbb{P}^{2M+1}$ intersects $\mathbb{P}(\Delta)$ if and only if $y = z$. Further, $(y\#y).\mathbb{P}(\Delta) = (y, y)$.*

PROOF Let $u \in k^{t+1}, v \in k^{t+1}$ be vectors representing $y, z$, i. e. $[u] = y, [v] = z$. The join $y\#z \subset \mathbb{P}^{2t+1}$ consists of the points $[(au, bv)]$, $a, b \in k$. If $[(au, bv)] \in \mathbb{P}(\Delta)$, then there is a vector $w \in k^{t+1}$ such that $(au, bv) = (w, w)$. Hence, $au = w = bv$, that is $y = [u] = [v] = z$, and $[(w, w)] = (y, y)$.

**3.13 Lemma** *Let $x$ be a closed point in projective space, $Y, Z$ properly intersecting effective cycles in $\mathbb{P}^M$, and $Y\#Z$ their join in $\mathbb{P}^{2M+1}$.*

1.
$$v_{(x,x)}(Y\#Z) \geq v_x(Y)v_x(Z).$$

2.
$$v_{(x,x)}(\delta_*(Y.Z)) = v_x(Y.Z),$$
*where $\delta : \mathbb{P}^M \times \mathbb{P}^M \to \mathbb{P}(\Delta)$ is the diagonal embedding.*

PROOF 1. By Fact 3.7.4, there are subspaces $\mathbb{P}(F), \mathbb{P}(F') \subset \mathbb{P}^t$ such that $v_x(Y) = v_x(\mathbb{P}(F).Y), v_x(Z) = v_x(\mathbb{P}(F').Z)$. Since $\mathbb{P}(F).Y = \sum_y n_y y$ is zero dimensional, by Lemma 3.7, $v_x(\mathbb{P}(F).Y) = n_x$ similarly, with $\mathbb{P}(F').Z = \sum_z m_z z$, the equality $v_x(\mathbb{P}(F').Z) = m_x$ holds. Since $(Y\#Z).(\mathbb{P}(F)\#\mathbb{P}(F')) = \sum_{y,z} n_y n_z y\#z$, and $x\#x$ contains $(x, x)$, it follows from the previous Lemma that

$$v_{(x,x)}(Y\#Z) = v_{(x,x)}(Y\#Z).(\mathbb{P}(F)\#\mathbb{P}(F')) \geq n_x m_x = v_x(Y)v_x(Z).$$

2. Since the diagonal embedding is an isomorphism, this follows from the previous Lemma.

PROOF OF THEOREM 3.9 By the previous Lemma, part one,

$$v_{(x,x)}(Y\#Z) \geq v_x(Y)v_x(Z).$$

Further, by Remark 3.10,

$$v_{(x,x)}(Y\#Z) \leq v_{(x,x)}(\mathbb{P}(\Delta).(Y\#Z)) = v_{(x,x)}(\delta_*(Y.Z)),$$

which by part 2 of the previous Lemma equals $v_x(Y.Z)$.

**3.14 Definition** *Let $\mathcal{Y}$ be an effective cycle in $\mathbb{P}^M_{\mathbb{Z}}$, and $Y$ its base extenseion to Spec $\mathbb{Q}$. For a real number $H$ the weighted order of vanishing of $\mathcal{Y}$ at a point $x$ in $\mathbb{P}^M_k$ is defined as $v_x(Y)/t_H(\mathcal{Y})$.*

**3.15 Lemma** *For every effective cycle $\mathcal{Y}$, and every closed point $x \in \mathbb{P}^M$, there is an irreducible component $\bar{\mathcal{Y}}$ of $\mathcal{Y}$ such that*

$$\frac{v_x(\bar{Y})}{t_H(\bar{\mathcal{Y}})} \geq \frac{v_x(Y)}{t_H(\mathcal{Y})}.$$

Proof Follows from the fact that both $v_x$ and $t_H$ are linear functions on $Z(\mathbb{P}^t)$, and elementary arithmetic.

**3.16 Proposition** *Let $X \subset \mathbb{P}^M$ be an irreducible subvariety of dimension $t$, and $\alpha$ a closed point in $X$. Further, $Y$ a subvariety of codimension $p$ in $X$ containing $\alpha$, and $f_i \in \Gamma(\mathbb{P}^M, O(D_i)), i = 1, \dots t-p$ global sections such that for every $i = 0, \dots, t-p$ there is an effective cycle $Z_i$ of pure codimension $i+p$ such that $Z_0 = Y$, the intersection of $Z_i$ with $divf_{i+1}$ is proper, and $Z_{i+1} + X_i = divf_{i+1}.Z_i$, where $X_i$ is an effective cycle whose support does not contain $\alpha$. Further, assume that for every $i = 1, \dots, t-p$ there is a number $S_i \in \mathbb{N}$ such that $\partial^I f_i$ is zero on $\alpha$ for every $i = 1, \dots, t-p$, $I$ with $|I| \leq S_i$, and $\partial^I$ a derivation of the functions field of $X$ as above. Then,*

$$v_\alpha(Z_{t-p)} \geq S_1 \cdots S_{t-p}.$$

Proof By fact 3.7.1, $v_\alpha(Y) \geq 1$, and by Proposition 3.8, the vanishing order of $f_i$ at $\alpha$ is at least $S_i$. Hence, by the local Bézout Theorem,

$$v_\alpha(Z_{i+1}) = v_\alpha(Z_{i+1} + X_i) = v_\alpha(\mathrm{div} f_{i+1}.Z_i) = v_\alpha((X.\mathrm{div} f_{i+1}).Z_i) \geq$$

$$v_\alpha(X) v_\alpha(\mathrm{div} f_{i+1}) v_\alpha(Z_i) \geq 1 S_{i+1} v_\alpha(Z_i),$$

and the Proposition follows by complete induction.

## 3.3 Weighted derivative algebraic distance

In analogy to the weighted algebraic distance defined in [Ma3], define the weighted derivated algebraic distance

**3.17 Definition** *Let $\mathcal{X}$ be an effective cycle in $\mathbb{P}^t$. The a-size of $\mathcal{X}$ is defined to be the number*

$$t_a(\mathcal{X}) := a \deg X + h(\mathcal{X}).$$

*For $S \in \mathbb{N}$ define the weighted derivated algebraic distance of $\mathcal{X}$ to $\theta$ as*

$$\varphi_a^S(\theta, \mathcal{X}) := \frac{D^{3S}(\theta, X)}{t_a(\mathcal{X})}.$$

**3.18 Lemma** *Let $\mathcal{X}$ be an effective cycle in $\mathbb{P}^t$, and $S$ a natural number. Then, there is an irreducible component $\mathcal{Y}$ of $\mathcal{X}$ and an $S_Y \in \mathbb{N}$ with $S_Y/t_a(\mathcal{Y}) \geq S/t_a(\mathcal{X})$ such that*

$$2\varphi_a^{S_Y}(\theta, \mathcal{Y}) \leq \varphi_a^S(\theta, \mathcal{X}) + O\left(\frac{\log \deg X}{a}\right).$$

*We call $\mathcal{Y}$ the irreducible component with minimal derivated algebraic distance relative to $S$.*

Proof Let $\mathcal{Y}, \mathcal{Z}$ be effective cycles of codimension $p$ in $\mathbb{P}^t$, and $S \in \mathbb{N}$. By [Ma4], Theorem 5.1, there are subspaces $\mathbb{P}(F), \mathbb{P}(F')$ of codimension $t-p$ such that with $y_1, \ldots y_{\deg Y}$ the points in the intersection of $\mathbb{P}(F)$ with $Y$ counted with multiplicity, and likewise $z_1, \ldots, z_{\deg Z}$ for $\mathbb{P}(F')$ and $Z$ for all natural numbers $S_1 \leq \deg Y/3, S_2 \leq \deg Z/3$,

$$D^{S_1}(Y, \theta) \leq \sum_{i=S_1+1}^{\deg Y} \log |y_i, \theta| + O((S_1 + \deg Y) \log(S_1 \deg Y)),$$

$$D^{S_2}(Z, \theta) \leq \sum_{i=s_2+1}^{\deg Z} \log |z_i, \theta| + O((S_2 + \deg Z) \log(S_2 \deg Z)).$$

For given $S$ choose $S_1, S_2 \in \mathbb{N}$ such that $S_1 + S_2 = S$, and

$$\log |y_i, \theta| \leq \log |z_j, \theta| \quad \text{for} \quad i \leq S_1, j \geq S_2,$$

$$\log |z_j, \theta| \leq \log |y_i, \theta| \quad \text{for} \quad j \leq S_2, i \geq S_1. \tag{5}$$

Then,

$$\sum_{i=S_1+1}^{\deg Y} \log |y_i, \theta| + \sum_{j=S_2+1}^{\deg Z} \log |z_j, \theta| \leq \inf_{\mathbb{P}(F)} \sum_{z \in supp(\mathbb{P}(F).(X+Z)} n_z \log |z, \theta| \leq$$

$$\frac{1}{2} D^{3S}(\theta, X+Y) + O(\deg(X+Y) \log(\deg(X+Y))),$$

again by [Ma4], Theorem 5.1. Consequently,

$$\frac{\sum_{i=S_1+1}^{\deg Y} \log |y_i, \theta| + \sum_{j=S_2+1}^{\deg Z} \log |z_j, \theta|}{t_a(\mathcal{X}+\mathcal{Y})} \leq \frac{1}{2} \frac{D^{3S}(\theta, Y+Z)}{t_a(\mathcal{Y}+\mathcal{Z})} + O\left(\frac{\log(\deg(Y+Z))}{a}\right).$$

Let $r \in \mathbb{R}$ be such that $\frac{S_1 + r}{t_a(\mathcal{X})} = \frac{S}{t_a(\mathcal{X}+\mathcal{Y})}$, and $s = \text{sign} r[|r|]$. Then, $|s| \leq \min(S - S_1, S - S_2)$, and by (5),

$$\frac{\sum_{i=S_1+1}^{\deg Y} \log |y_i, \theta| + \sum_{j=S_2+1}^{\deg Z} \log |z_j, \theta|}{t_a(\mathcal{Y}+\mathcal{Z})} \geq$$

$$\frac{\sum_{i=S_1+s+1}^{\deg Y} \log |y_i, \theta| + \sum_{j=S_2-s+1}^{\deg Z} \log |z_j, \theta|}{t_a(\mathcal{Y}) + t_a(\mathcal{Z})}.$$

By elementary arithmetic, this is greater or equal

$$\min\left(\frac{\sum_{i=S_1+s+1}^{\deg Y} \log |y_i, \theta|}{t_a(\mathcal{Y})}, \frac{\sum_{j=S_2-s+1}^{\deg Z} \log |z_j, \theta|}{t_a(\mathcal{Z})}\right).$$

Further

$$\frac{S_1 + r}{t_a(\mathcal{X})} = \frac{S_2 - r}{t_a(\mathcal{Y})} = \frac{S}{t_a(\mathcal{X} + \mathcal{Y})}.$$

By complete induction it follows, that for any effective cycle $\mathcal{X}$ with decomposition into irreducible parts

$$\mathcal{X} = \sum_{k=1}^{n} \mathcal{X}_1,$$

we have numbers $S_k$ with $S_k/t_a(\mathcal{X}_k) = S/t_a(\mathcal{X}) + \epsilon$, and

$$\min_{k=1,\dots n}\left(\frac{\sum_{i=S_k+1}^{\deg X_k} |\log x_{ik}, \theta|}{t_a(\mathcal{X}_k)}\right) \leq \frac{D^{3S}(\theta, X)}{2t_a(\mathcal{X})} + O((\log \deg X)/a).$$

The Lemma follows by once more using [Ma4], Theorem 5.1.

**3.19 Lemma** *Let $\mathcal{Y} \in Z_{eff}(\mathbb{P}^M)$ an effective cycle, and $\theta \in \mathbb{P}^M(\mathbb{C})$ a point not contained in the support of $\mathcal{Y}$. Then, for any $m, n, S \in \mathbb{N}$.*

$$D^{nS}(\theta, mnY) \leq mD^{nS}(\theta, Y).$$

Proof Since

$$\exp(D(\theta, mnY)) = (\exp(D(\theta, X)))^{mn},$$

this follows by elmentary differentiation techniques.

# 4 Projection to a projective sub space

**4.1 Proposition** *Let $X \subset \mathbb{P}^M_{\mathbb{C}}$ be a subvariety of dimension $t$, further $\theta \in X(\mathbb{C})$, and $Y$ an effective cycle in $X$ whose support does not contain $\theta$. Let $\varphi : \mathbb{A}^M(\mathbb{C}) \to \mathbb{P}^M(\mathbb{C})$ be an affine chart centered at $\theta$ such that $\varphi(\mathbb{A}^t \times \{0\} = \mathbb{P}(T_\theta X)$ the tangent space of $X$ at $\theta$. Denote by $I$ a multi index, and by $\partial^I$ the corresponding differential. Further let $N_t$ be the set of multi indizes $I = (1_1, \dots, i_{2M})$ with $i_{2t+1} = \cdots = i_{2M} = 0$. Then, for $S \leq \deg Y/3$,*

$$\sup_{I \in N_t, |I| \leq S} \log \left|(\partial^I(\varphi^* \exp D(Y, \theta))\right| \leq D^S(\theta, Y).$$

$$D^S(\theta,Y)\le \sup_{I\in N_t,|I|\le S}\log\left|(\partial^I(\varphi^*\exp D(Y,\theta))\right|+O(\deg Y\log\deg Y).$$

PROOF The first claim is trivial. For the second claim, let $U_\theta$ be a neighbourhood of $\theta$ in $X$ such that the orthogonal projection $\pi$ of $U_\theta$ to $T_\theta X$ is bijective, and for every $x\in U_\theta$, the inequality $|x,\theta|\le 2|\pi x,\theta|$ holds. With $p$ the codimension of $Y$ in $X$, by [Ma4], Theorem 1.4, there is a subspace $\mathbb{P}(F)\subset\mathbb{P}^M$ of codimension $t-p$ such that $\mathbb{P}(F)$ contains $\theta$, intersects $Y$ properly, and with $\mathbb{P}(F).Y=\sum_{i=1}^{\deg Y}y_i$, numbered in such a way that $|y_1,\theta|\le\cdots\le|y_{\deg Y},\theta|$ the derivated algebraic distance of $\theta$ to $Y$ may be estimated as

$$D^S(\theta,Y)\le\sup_{|I|\le S}\log|\partial^I\prod_{i=1}^{\deg Y}|y_i,\theta||+O(S\log\deg Y),$$

$$\log|\partial^I\prod_{i=1}^{\deg Y}|y_i,\theta||\le D^S(\theta,Y)+O(\deg Y).$$

Let $r$ be the radius of $U_\theta$, and $k\le\deg Y$ a number such that $|y_k,\theta|\le r\le|y_{k+1},\theta|$. Then, with $c_i(z)=|\pi y_i,\theta|/|y_i,\theta|$,

$$\log|(\partial^I\varphi^*c_i(z))(0)|\le c|I|,\quad \log|(\partial^I 1/c_i(z))(0)|\le c|I|,$$

with $c$ a fixed constant. Hence, for every $I$, with $|I|\le S$,

$$\log\sup_{|I|\le S}|\partial^I\prod_{i=1}^{\deg Y}|y_i,\theta||\le\log\sup_{|I|\le S}|\partial^I\prod_{i=1}^{\deg Y}|\pi y_i,\theta||+cS\le$$

$$\log\sum_{I\in N_t,|I|\le S}\partial^I\prod_{i=1}|\pi y_i,\theta||+cS\le\log\sup_{I\in N_t,|I|\le S}|\partial^I(\varphi^*\exp D(Y,\theta))|,$$

proving the second claim.

**4.2 Lemma** *There are positive constants $\bar c,\tilde c$ only depending on $M$ and $t$, and a subspace $\mathbb{P}^{M-t-1}\subset\mathbb{P}^M$ defined over $\mathbb{Z}$ that does not meet $\mathcal{X}$, and fulfills*

$$h(\mathbb{P}^{M-t-1})\le\tilde c\log\deg X\quad\textit{and}\quad\log|\mathbb{P}^{M-t-1},X|\ge-\bar c-\log\deg X.$$

*For $\mathbb{P}^t$ the orthogonal complement of $\mathbb{P}^{M-t-1}$ in $\mathbb{P}^M$, the restriction of the map*

$$\pi:\mathbb{P}^M\setminus\mathbb{P}^{M-t-1},\quad [v+w]\mapsto[v],[v]\in\mathbb{P}^t,[w]\in\mathbb{P}^{M-t-1}$$

*to $\mathcal{X}$ is a flat, finite proper map $\pi_X:\mathcal{X}\to\mathbb{P}^t$, and*

$$h(\mathbb{P}^t)\le c\log\deg X,$$

*with $c$ a constant only depending on $M$ and $t$.*

Proof By [Ma4], Corollary 5.4, there is a subspace $\mathbb{P}(W) \subset \mathbb{P}^M_{\mathbb{C}}$ with

$$\log |\mathbb{P}(W), X| \geq -c_1 - \log \deg X,$$

with some positive constant $c_1$ only depending on $M$, and $t$. For $V \subset \mathbb{C}^{M+1}$ a subspace, denote by $S(V)$ the set of vectors of length 1 in $V$, and by $pr_V^\perp$ the orthogonal projection to the orthogonal complement of $V$. On the Grassmannian $G_{M,t,}$ we have

$$|V, W| = \sup_{v \in S(V)} |pr_W^\perp|,$$

and for $V$ a primitive submodule of $\mathbb{Z}^{t+1}$,

$$h(V) = \log \operatorname{vol}(V) + \sigma_p.$$

Let $W$ be the space from above, $q = M - t = \dim W$, and $a = 2e^{c_1} q(t+1) \deg X$. One can recursively find vectors

$$v_1, \ldots, v_q \in \mathbb{C}^{M+1},$$

such that with $V_i = \langle v_i, \ldots, v_i \rangle$,

$$v_i \in pr_{V_{i-1}}^\perp(\mathbb{Z}^{M+1}), \quad |v_i| \leq (M+1) 2^{(M+1)/q} a^{t+1}, \quad |pr_W^\perp(v_i)| \leq \sqrt{t+1}\frac{1}{a}.$$

Indeed, assume that $w_1, \ldots, w_{t+1}$ is an orthonormal basis of $W^\perp$, and $v_1, \ldots, v_i$ have been found. Since, $\log \operatorname{vol} V_i \geq 1$, then $\log \operatorname{vol}(\mathbb{Z}^{M+1}/V_i)) \leq 1$. Let $Q$ be the Cuboid in $\mathbb{R}^{t+1}$ that has lengths $2^{(M+1)/(t+1)} a^{q/(t+1)}$ parallel to $W$, and lengths $1/a$ paralell to $W^\perp$. Then,

$$\operatorname{vol}(Q) = 2^{M+1} a^{t+1} (1/a)^{t+1} = 2^{M+1} \geq 2^{M+1} \operatorname{vol}(\mathbb{Z}^{M+1}/V_i).$$

By the Theorem of Minkovksi, $Q$ thus contains a non zero vector $v_{i+1}$, and $v_{i+1}$ fulfills

$$|v_{i+1}|^2 \leq q(2^{(M+1)/(t+1)} a^{q/(t+1)})^2 + (t+1)(1/a)^2 \leq (M+1) 2^{(M+1)/(t+1)} a^{2q/(t+1)},$$

and

$$|pr_W^\perp(v_{i+1})|^2 \leq (t+1) \left(\frac{1}{a}\right)^2.$$

Since $v_1, \ldots, v_q$ is an orthonormal basis of $V = V_q$, for any $v \in S(V)$ we have $v = \sum_{i=1}^q a_i v_i$ with $\sum_{i=1}^q |a_i|^2 = 1$, hence

$$|pr_W^\perp(v)| \leq \sum_{i=1}^q a_i |pr_W^\perp(v_i)| \leq q\sqrt{t+1}\,\frac{1}{a} = \frac{q(t+1)}{2e^{c_1} q(t+1) \deg X} = \frac{e^{-c_1}}{2 \deg X}.$$

Hence,

$$\log|V,W| = \log \sup_{v\in S(V)} |pr_W^\perp(v)| \leq -c_1 - \log 2 - \log\deg X.$$

Since $\log|W,X \geq -c_1 - \log\deg X$, we get $\log|V,X| \geq -c_1 - \log\deg X - \log 2 = -\bar{c} - \log\deg X$ with a suitable $\bar{c}$.
Finally, since $|v_i| \leq (M+1)2^{(M+1)/q} a^{t+1}$ for $i = 1,\dots q$, we get

$$h(\mathbb{P}(V)) = \log\mathrm{vol}(V) + \sigma_q \sum_{i=1}^{q} \log|v_i| + \sigma_q \leq$$

$$\log\left(q(M+1)2^{(M+1)/(t+1))}(2e^{c_1}q(t+1)\deg X)^{q/(t+1)}\right) + \sigma_q \leq \tilde{c}\log\deg X,$$

with a suitable $\tilde{c} > 0$. If $M = \mathbb{Z}^{M+1} \cap V$, and $M^\perp = \mathbb{Z}^{t+1} \cap V^\perp$, by [Be], Proposition 1.(ii),

$$\mathrm{vol}M^\perp \mathrm{vol}M \leq (\deg X)^{\tilde{c}} / \exp(\sigma_q).$$

Hence, with $\mathbb{P}^t = \mathbb{P}(M^\perp)$,

$$h(\mathbb{P}^t) = \log\mathrm{vol}M^\perp + \sigma_t \leq \tilde{c}\log\deg X + \sigma_t - \sigma_q \leq c\log\deg X.$$

**4.3 Proposition** *Let $\mathcal{Y} \in Z^p_{eff}(\mathcal{X})$ be a cycle, $\theta \in X(\mathbb{C})$ a generic point, and $\mathbb{P}^t, \mathbb{P}^{M-t-1}, \pi, \pi_X$ as in Lemma 4.2*

1. *If the set of complex valued points $Y_i$ of an irreducible component $\mathcal{Y}_i$ of $\mathcal{Y}$ has sufficiently small distance to $\theta$, then $\dim\pi(\mathcal{Y}_i) = \dim\mathcal{Y}_i$.*
2. *For $x, y \in X(\mathbb{C})$ in a sufficiently small neighbourhood of $\theta$,*

$$|x,y| \leq c|\pi_X x, \pi_X y|,$$

*where $c$ is constant depending on $\theta$. and for $x, y \in \mathbb{P}^M \setminus \mathbb{P}(F^\perp)$,*

$$\log|\pi x, \pi y| \leq |x,y| - \log|x, \mathbb{P}^{M-t-1}| - \log|y, \mathbb{P}^{M-t-1}|.$$

3. *If $Y$ is irreducible, $\dim\pi Y = \dim Y$, then*

$$\deg\pi_X(Y) = \deg Y, \quad h(\pi_X(\mathcal{Y})) \leq h(\mathcal{Y}).$$

4. *If $Y$ is irreducible, $\dim\pi Y = \dim Y$, $\theta$ is not contained in the support of $Y$, and $S \leq \deg Y/3$, then*

$$2(D^{\mathbb{P}^t})^S(\pi\theta, \pi_X(Y)) \leq D^{3S}(\theta, Y) + O(\deg Y\log\deg Y),$$

Proof 1. Let $T_\theta X$ be the tangent space of $\mathcal{X}$ at $\theta$ which may be identified with the projective space $\mathbb{P}(V_\theta)$ corresponding to a subspace $V_\theta$ of $\mathbb{C}^{M+1}$. Since $\theta$ is a generic point of $\mathcal{X}$, the restriction of $\pi$ to $\mathbb{P}(V_\theta)$ is bijective, hence comes from a bijective linear map

$$\varphi : V_\theta \to \mathbb{C}^{t+1}.$$

Because the metrics on $\mathbb{P}(V_\theta)$ and $\mathbb{P}^t$ just correspond to different inner products on $V_\theta$ and $\mathbb{C}^t$, there is positive constant $c$ such that

$$\frac{1}{c}|\pi x, \pi y| \leq |x, y| \leq c|\pi x, \pi y|$$

for every $x, y \in \mathbb{P}(V_\theta)$. Further, for a sufficiently small neighbourhood $U_\theta$ of $\theta$ the orthogonal projection $pr$ from $U_\theta$ to $T_\theta X = \mathbb{P}(V_\theta)$ is bijective, and

$$|prx, pry| \leq |x, y| \leq 2|prx, pry|$$

for every $x, y$ in $U_\theta$ implying the first claim.
Let $u, v \in \mathbb{C}^{t+1}$ be vectors representing $\pi x$ and $\pi y$. There are vectors $w_1, w_2 \in \mathbb{C}^{M-t}$ such that $\bar{u} = u + w_1, \bar{v} = v + w_2$ represent the points $x, y$. We may assume that $|\bar{u}| = |\bar{v}| = 1$. Then, in the Fubini-Study metric, since $u, w \in \mathbb{C}^{t+1}$, and $w_1, w_2 \in \mathbb{C}^{M-t} = (\mathbb{C}^{t-1})^\perp$,

$$|x, \mathbb{P}^{M-t-1}|^2 \leq |x, [w_1]|^2 = \sin^2(u, w_1) = |u|^2, \quad |y, \mathbb{P}^{M-t-1}| \leq |y, [w_2]|^2 = |v|^2.$$

Without loss of generality, we may assume $\langle u|v\rangle \leq 0$, and $|u| \leq |v|$, hence $|w_2| \leq |w_1|$. If $|w_2| = 0$, then $x = \pi x$, $y = \pi y$, and there is nothing to prove. If $|w_2| > 0$, set $\lambda = |w_2|/|w_1| \leq 1$, and define the point $\tilde{y} \in \mathbb{P}^M$ by $y = [v + \lambda w_1]$.
Then,

$$|x, y|^2 = 1 - (\langle u|v\rangle + \langle w_1|w_2\rangle)^2 \geq 1 - (\langle u|v\rangle + |w_1||w_2|)^2 =$$
$$1 - (\langle u|v\rangle + \lambda\langle w_1|w_1\rangle)^2 = |x, \tilde{y}|^2.$$

Further,

$$|u|^2|v|^2|\pi x|\ |\pi y|^2 = |u|^2|v|^2(1 - \langle u|v\rangle^2) \leq$$
$$|u|^2 + |v|^2 - |u|^2|v|^2 - 2\langle u|v\rangle|w_1||w_2| - \langle u|v\rangle^2 =$$
$$1 - (1-|u|^2)(1-|v|^2) - \langle u|v\rangle^2 - 2\langle u|v\rangle|w_1||w_2| = 1 - \langle u|v\rangle^2 - 2\langle u|v\rangle\lambda|w_1|^2 - \lambda^2|w_1|^4 =$$
$$1 - \langle u|v\rangle + \langle w_1|\lambda w_2\rangle^2 = |x, \tilde{y}|,$$

which, together with the above, implies

$$|x, \mathbb{P}^{M-t-1}|^2|y, \mathbb{P}^{M-t-1}|^2|\pi x|\pi y|^2 \leq |x, y|.$$

2. Since $\theta$ is a generic point, the base extension $\pi_\mathbb{C}$ to $X_\mathbb{C}$ is injective in some neighbourhood of $\theta$. This immediately implies the claim.

3. The first claim is obvious. The second claim is [BGS], (3.3.7).

4. Let $p$ be the dimension of $Y$. Since $|\mathbb{P}^{M-t-1}, X| \geq -\bar{c} - \log \deg X$, by [Ma4], Propositions 5.4, and Corollary 5.5, there is a space $\mathbb{P}(F) \subset \mathbb{P}^M$ of codimension $t-p$ that contains $\mathbb{P}^{M-t-1}$ as well as $\theta$, hence intersects $Y$ properly, such that

$$(D^{\mathbb{P}(F)})^S(\theta, Y.\mathbb{P}(F)) \leq D^S(\theta, Y) + O(\deg Y \log \deg Y),$$

hence, if $\mathbb{P}(F).Y = \sum_{i=1}^{degY} y_i$ where the $y_i$ are ordered in such a way that $|y_1, \theta| \leq \cdots \leq |y_{\deg Y}, \theta|$, [Ma4], Proposition 4.7 implies

$$2 \sum_{i=S+1}^{\deg Y} \log |y_i, \theta| \leq D^{3S}(\theta, Y) + O(\deg Y \log \deg Y).$$

Let $\sigma \in \Sigma_{\deg Y}$ be a permutation such that $|\pi y_{\sigma 1}, \pi\theta| \leq \cdots \leq |\pi y_{\sigma \deg Y}, \theta|$. By part Proposition 4.5, $|\pi y_i, \pi\theta| \leq |y_i, \theta| + c \log \deg Y$. Hence,

$$2 \sum_{i=S+1}^{\deg Y} \log |\pi y_{\sigma i}, \pi\theta| \leq 2 \sum_{i=S+1}^{\deg Y} \log |y_{\sigma i}, \theta| + c(\deg Y - S) \log \deg Y \leq$$

$$2 \sum_{i=S+1}^{\deg Y} \log |y_i, \theta| + c(\deg Y - S) \log \deg Y \leq D^{3S}(\theta, Y) + O(\deg Y \log \deg Y).$$

Further, since $\mathbb{P}(F) \cap \mathbb{P}^t$ is a subspace of dimension $p$ in $\mathbb{P}^t$ containing $\pi\theta$ and intersecting $\pi Y$ properly, [Ma4], Proposition 5.1 implies

$$(D^{\mathbb{P}^t})^S(\pi\theta, \pi Y) \leq \sum_{i=S+1}^{\deg Y} \log |\pi y_{\sigma i}, \pi\theta| + O(\deg Y \log \deg Y),$$

hence

$$2(D^{\mathbb{P}^t})^S(\pi\theta, \pi Y) \leq D^{3S}(\theta, Y) + O(\deg Y \log \deg Y),$$

as was to be proved.

**4.4 Lemma** *Let $\mathcal{Y} \in Z^p_{eff}(\mathbb{P}^M)$ be an effective cycle that intersects $\mathcal{X}$ properly, and $\theta \in X(\mathbb{C})$ a point not contained in the support of $Y$. Then,*

1.
$$\deg(X.Y) = \deg X \deg Y,$$
$$h(\mathcal{Y}) \leq \deg h(\mathcal{Y}) + \deg Y h(\mathcal{X}) + c \deg X \deg Y.$$

2. *For any $S \leq \deg Y$,*

$$2D^S(\theta, Y.X) \leq D^(3S)(\theta, Y) + O(\deg X \deg Y \log(\deg X \deg Y)).$$

PROOF 1. is just the algebraic and arithmetic Bézout Theorem. Since $\theta \in X(\mathbb{C})$, 2. is Theorem 2.9.2 applied to the varieties $X, Y$.

**4.5 Proposition** *In the situation of Lemma 4.2, let $\mathcal{Y} \in Z^p_{eff}(\mathbb{P}^t)$. Then, $\mathcal{X}$ intersects $\pi^*(\mathcal{Y})$ properly, and $\mathcal{Y}^* := \pi^*_{\mathcal{X}}(\mathcal{Y}) = \pi^*(\mathcal{Y}).\mathcal{X}$. Further,*

1.
$$\deg Y^* = \deg X \deg Y,$$
$$h(\mathcal{Y}^*) \leq \deg X(h(\mathcal{Y}) + \tilde{c} \deg Y \log \deg X) + \deg Y h(\mathcal{X}) + c \deg X \deg Y,$$
*and for every irreducible component $\bar{\mathcal{Y}}^*$ of $\mathcal{Y}^*$ sufficiently close to $\theta$,*
$$\deg \bar{Y}^* \geq \deg Y, \quad h(\bar{\mathcal{Y}}^* \geq h(\mathcal{Y}).$$

2. *If further $\theta \in \mathbb{P}^t(\mathbb{C})$ is not contained in the support of $Y$, and $\bar{\theta} \in \mathcal{X}(\mathbb{C})$ is a point with $\pi_X \bar{\theta} = \theta$, then for $S \leq \deg Y$,*
$$D^S(\bar{\theta}, Y^*) \leq \frac{1}{4} D^{9S}(\theta, Y) + \deg X h(\mathcal{Y}^*) + \deg Y^* h(\mathcal{X}) + d \deg X \deg Y^*.$$

3. *If $f \in \Gamma(\mathbb{P}^t, O(D))$, let $f^* \pi^* f$. Then,*
$$\log |f^*|_{L^2(\mathbb{P}^M)} = |f|_{L^2(\mathbb{P}^t)} + cD,$$
$$|divf^*, \theta| \leq c|divf, \pi\theta| \leq c|divf^*, \theta| + cc_2 \deg X.$$
$$\sup_{|I| \leq S} \log |(\partial^I f^*)(\theta)| \leq \sup_{|I| \leq S} \log |(\partial^I f)(\bar{\theta})|.$$

PROOF
1. Since $\deg \pi^* Y = \deg Y$, the first claim follows from $\pi^*_{\mathcal{X}}(\mathcal{Y}) = \pi^*(\mathcal{Y}).\mathcal{X}$ and the previous Lemma.
Let $x_1, \ldots, x_{M-t} \in \Gamma(\mathbb{P}^M, O(1))$ such that $\mathbb{P}^t = \mathrm{div} x_1 . \ldots . \mathrm{div} x_{M-t}$. Then, by Lemma 2.1,
$$\sum_{i=1}^{M-t} \int_{\mathrm{div} x_1 . \ldots . \mathrm{div} x_{i-1}} \log |x_i| \mu^{M-i} = h(\mathbb{P}^t) - h(\mathbb{P}^M),$$
and $\mathcal{Y} = \pi^*(\mathcal{Y}).\mathrm{div} x_1 . \ldots . \mathrm{div} x_{M-t}$. Hence, there are numbers $a_1, \ldots, a_{M-t} \in \mathbb{R}$ such that $\sum_{i=1}^{M-t} a_i = h(\mathbb{P}^t) - h(\mathbb{P}^M)$, and $\log |x_i| - a_i$ is a normalized Green form for $\mathrm{div} x_1 . \ldots . \mathrm{div} x_i$ in $\mathrm{div} x_1 . \ldots . \mathrm{div} x_{i-1}$. The equality $\mathcal{Y} = \pi^*(\mathcal{Y}).\mathrm{div} x_1, \ldots . \mathrm{div} x_{M-t}$ together with Lemma 2.1 and [BGS], Proposition 5.1 implies
$$\begin{aligned} h(\mathcal{Y}) - h(\pi^*(\mathcal{Y})) &= \sum_{i=1}^{M-t} \int_{\pi^*(Y).\mathrm{div} x_1 . \ldots . \mathrm{div} x_{i-1}} \log |x_i| \mu^{m-p-i} \\ &= \sum_{i=1}^{M-t} \int_{\pi^*(Y).\mathrm{div} x_1 . \ldots . \mathrm{div} x_{i-1}} (\log |x_i| - a_i) \mu^{M-p-j} + \deg Y \sum_{i=1}^{M-t} a_i \\ &= -c \deg Y + \deg Y (h(\mathbb{P}^t) - h(\mathbb{P}^M), \end{aligned}$$

with $c$ a positive constant depending only on $t, M$, and $p$. Thus,

$$h(\pi^*(\mathcal{Y})) = h(\mathcal{Y}) + c\deg Y - \deg Y(h(\mathbb{P}^M) - h(\mathbb{P}^t)) \leq$$

$$h(\mathcal{Y}) + c\deg Y + c_1 \deg Y \log\deg X.$$

Since $\pi^*_{\mathcal{X}}(\mathcal{Y}) = \pi^*(\mathcal{Y}).\mathcal{X}$, the previous Lemma implies

$$h(\pi^*_{\mathcal{X}}(\mathcal{Y})) \leq \deg X h(\pi^*(\mathcal{Y})) + \deg Y h(\mathcal{X}) + c_2 \deg X \deg Y \leq$$

$$\deg X(h(\mathcal{Y}) + c_1 \deg Y \log\deg X) + \deg Y h(\mathcal{X}) + c_3 \deg X \deg Y,$$

proving the second claim.
If $\bar{\mathcal{Y}}^*$ is an irreducible component of $\mathcal{Y}^*$ sufficiently close to $\theta$, then because of the irreduciblity, $(\pi_X)_*\bar{\mathcal{Y}}^* = \mathcal{Y}$, hence by Proposition 4.3.2, $\deg \bar{Y}^* \leq \deg Y, h(\bar{\mathcal{Y}}^*) \leq h(\mathcal{Y})$.

2. Let $U_\theta$ be a sufficiently small neighbourhood of $\theta$ in $X(\mathbb{C})$.
By [Ma4], Theorem 1.4, there is a subspace $\mathbb{P}(F) \subset \mathbb{P}^t$ of dimension $p$ such that with $\mathbb{P}(F).Y = \sum_{i=1}^{\deg Y} y_i$, ordered such that $|y_1, \theta| \leq \cdots \leq |y_{\deg Y}, \theta|$,

$$2\sum_{S+1}^{\deg Y} \log|y_i, \theta| \leq D^{3S}(\theta, Y) + O((S + \deg Y)\log\deg Y),$$

for every $S \leq \deg Y/3$. Denote by $l \leq \deg Y$ the number such that $y_i \in \pi_X U_\theta$ for $i \leq l$, and $y_i \notin \pi U_\theta$ for $i > l$. Then, $\log|y_i, \theta| \geq -c_2$ for every $i > l$ with $c_2 > 0$ independent of $Y$. Further, let $\mathbb{P}(F^*) \subset \mathbb{P}^M$ be the projective subspace of codimension $p$ that Contains $\mathbb{P}(F)$ as well as $\mathbb{P}^{M-t-1}$. Then the restriction of $\pi_X$ to $U_\theta$ maps $\mathbb{P}(F^*) \cap \mathrm{supp}(\pi^*(Y))$ injectively to $\mathbb{P}(F \cap \mathrm{supp} Y)$, and since $|\bar{\theta}, \mathbb{P}^{M-t}| \geq c\deg X$, for every $y^*$ in $\mathbb{P}(F^*) \cap \pi^*(Y)$, we have $\log|y^*, \bar{\theta}| \leq \log|\pi(y^*), \theta| + c_1 \log\deg X$, and consequently if $\mathbb{P}(F^*).\pi^*(Y) = \sum_{i=1}^{\deg Y} y_i^*$ ordered in the usual way,

$$\begin{aligned}
D^S(\pi^* Y, \theta) &\leq \sum_{i=S+1}^{\deg Y} \log|y_i^*, \theta| + O(S\log\deg Y) \\
&\leq \sum_{i=S+1}^{l} \log|y_i^*, \theta| + O(S\log\deg Y) \\
&\leq \sum_{i=S+1}^{l} \log|y_i, \theta| + c_1 \deg Y \log\deg X + O(S\log\deg Y) \\
&\leq \sum_{i=S+1}^{\deg Y} \log|y_i, \theta| + (c_2 + c_1)\deg Y \log\deg X + O(S\log\deg Y).
\end{aligned}$$

Hence,

$$D^S(\pi^*(Y),\bar{\theta}) \leq \frac{1}{2}D^{3S}(Y,\theta) + (c_2+c_1)\deg Y \log\deg X + O(S\log\deg Y).$$

3. The first claim follows by integration over the fibres of $\pi$, and the second claim from Proposition 4.5.1.
With $\varphi : \mathbb{A}^t \to \mathbb{P}^t$ the canonical affine chart centered at $\theta$, and $\psi$ the local inverse of $\pi_X$ at $\bar{\theta}$ with image in $U.\theta$, the map $\psi \circ \varphi$ is an affine chart of $\mathcal{X}$ around $\theta$. Thus, for an $f \in \Gamma(\mathbb{P}^t, O(D))$,

$$(\psi \circ \varphi)^* \circ \pi^* f = \varphi^* f,$$

from which the claim about derivatives follows.
The inequality $|\mathrm{div} f^*, \theta| \leq |\mathrm{div} f, \pi\theta| \leq |\mathrm{div} f^*, \theta| + c\deg X$ follows from part 1.

# 5 Proof of second criterion

This section establishes a proof of Theorem 1.2. For a given $a > 1$, if $H_k \leq aD_k$, one can replace $H_k$ by $\bar{H}_k = aD_k$. Then, since $\bar{H}_k + D_k \leq (a+1)D_k \leq (a+1)(D_k + H_k)$, still

$$\limsup_{k\to\infty} \frac{S_k^s V_k}{D_k^s(D_k + \bar{H}_k)} = \infty,$$

hence we may from now on assume that $H_k \geq aD_k$. For similar reasons, one may assume $S_k \leq 3D_k$ for all $k$. Similarly, by replacing the series $(D_k, H_k, S_k, V_k)$ by $(5D_k, 5H_k, S_k, V_k)$, and each $f \in \mathcal{F}_k$ by $f^5$, one may assume that

$$\sup_{|I|\leq S_k - 1} |\log|\partial^I f|| \leq -5V_k$$

for each $k$ sufficiently big and $f \in \mathcal{F}_k$.

**5.1 Definition** *Given the series $(D_k, H_k, S_k, V_k)$, and a $t \leq s-1$, an irreducible subvariety $\mathcal{Y}$ of $\mathcal{X}$ of codimension $p \leq t$ is called sufficiently approximating of order $k$ and multiplicity $S_Y \in \mathbb{N}$ with respect to $\theta \in \mathcal{X}(\mathbb{C})$, if the estimates*

$$t_{H_k/D_k}(\mathcal{Y}) \leq \frac{S_Y}{S_k^p} 4^p D_k^{p-1} H_k, \tag{6}$$

*and*

$$\varphi_{H_k/D_k}^{S_Y/9^p}(\theta, \mathcal{Y}) \leq -\frac{4S_Y V_k}{14^{p-1} t_{\frac{H_k}{D_k}}(\mathcal{Y}) S_k} \tag{7}$$

*hold.*

**5.2 Lemma** *Given the series* $(D_k, H_k, S_k, V_k)$*, let* $C >> 0$ *and* $t \leq s-1$*. Because of*

$$\limsup_{k\to\infty} \frac{S_k^s V_k}{D_K^s(D_k+H_k)} = \infty$$

*for every* $k_0 \in \mathbb{N}$ *there is a* $k \geq k_0$ *such that*

$$\frac{S_k^s V_k}{D_k^s(D_k+H_k)} \geq C, \tag{8}$$

*and assume* $l \leq k$*.*

1. *Let* $\mathcal{Y}$ *be an irreducible subvariety of codimension* $p$ *in* $\mathcal{X}$*, and* $S_Y \in \mathbb{N}$ *a number such that (6) holds. Let further* $f \in \mathcal{F}_l$ *be such that* $divf$ *intersects* $\mathcal{Y}$ *properly, and assume*

   $$D^{S_Y(S_l-1)/9^{p+1}}(divf.Y,\theta) \leq -\frac{4S_lS_YV_k}{14^{p-1}S_k}.$$

   *Then, if either* $k=l$ *or* $|divf,\theta| \leq |Y,\theta|$*, there exists an irreducible component* $\bar{\mathcal{Y}}$ *of* $divf.\mathcal{Y}$ *and a number* $S_{\bar{Y}}$ *such that* $S_{\bar{Y}}/t_a(\bar{\mathcal{Y}}) \geq S_Y/t_a(divf.\mathcal{Y})$*, and* $\bar{\mathcal{Y}}$ *is sufficiently approximating of order* $k$ *and multiplicity* $S_{\bar{Y}}$ *with respect to* $\theta$*.*

2. *Let* $\mathcal{Y}$ *be an irreducible subvariety of codimension* $p$ *in* $\mathcal{X}$ *that is sufficiently approximating of order* $k$ *and multiplicity* $S_Y$ *with respcet to* $\theta$*, and* $f \in \mathcal{F}_k$ *a global section whose restriction to* $\mathcal{Y}$ *is nonzero. Then, there exists an irreducible component* $\bar{\mathcal{Y}}$ *of* $divf.\mathcal{Y}$*, and a number* $S_{\bar{Y}} \in \mathbb{N}$ *such that* $S_{\bar{Y}}/t_a(\bar{\mathcal{Y}}) \geq S_Y/t_a(divf.\mathcal{Y})$*, and* $\bar{\mathcal{Y}}$ *is sufficiently approximating of order* $k$ *and multiplicity* $S_{\bar{Y}}$ *with respect to* $\theta$*.*

Proof 1. Since

$$\varphi_{H_k/D_k}^{S_Y(S_l-1)/9^{p+1}}(\theta,\mathrm{div}f.\mathcal{Y}) \leq -\frac{4S_lS_YV_k}{14^{p-1}t_{H_k/D_k}(\mathrm{div}f.\mathcal{Y})S_k},$$

Lemma 3.18 implies that there is an irreducible component $\bar{\mathcal{Y}}$ of div$f.\mathcal{Y}$, and a number $S_{\bar{Y}}$ such that,

$$\varphi_{H_k/D_k}^{S_{\bar{Y}}}(\bar{\mathcal{Y}},\theta) \leq \varphi_{H_k/D_k}^{S_Y(S_l-1)}(f.\mathcal{Y},\theta) + O(\log(D_k \deg Y)) \leq -\frac{4\cdot S_lS_YV_k}{4\cdot 14^{p-1}t_{H_k/D_k}(\mathrm{div}f.\mathcal{Y})S_k},$$

and by shrinking $S_{\bar{Y}}$ if necessary,

$$2\,\frac{S_YS_l}{t_{H_k/D_k}(\mathrm{div}f.\mathcal{Y})} \geq S_{\bar{Y}}/t_{H_k/D_k}(\bar{\mathcal{Y}}) \geq \frac{S_YS_l}{t_{H_k/D_k}(\mathrm{div}f.\mathcal{Y})}. \tag{9}$$

Thereby,

$$\varphi^{S_{\bar{Y}}}_{H_k/D_k}(\bar{Y},\theta) \leq -\frac{4S_{\bar{Y}}V_k}{14^p t_{H_k/D_k}(\mathcal{Y})S_k}. \tag{10}$$

Further, by the algebraic and arithmetic Bézout Theorems, the inequality $D_l < H_k$, and the fact that $\mathcal{Y}$ fulfills (6),

$$\begin{aligned}
t_{H_k/D_k}(\mathrm{div} f.\mathcal{Y}) &\leq D_l h(\mathcal{Y}) + H_l \deg Y + \left(\frac{H_k}{D_k} + c\right) D_l \deg Y \\
&\leq 2D_l t_{H_k/D_k}(\mathcal{Y}) + 2H_l \frac{D_k}{H_k} t_{H_k/D_k}(\mathcal{Y}) \\
&\leq \frac{2S_Y}{S_k^p} 4^p D_l D_k^{p-1} H_k + \frac{2S_Y}{S_k^p} \frac{D_k}{H_k} 4^p H_l D_k^{p-1} H_k.
\end{aligned}$$

Hence, by the right hand side inequality of (9),

$$\frac{S_{\bar{Y}}}{t_{H_k/D_k}(\bar{\mathcal{Y}})} \geq \frac{S_l S_k^p}{2\cdot 4^p D_l D_k^{p-1} H_k + 2 \cdot 4^p D_k^p H_l} \geq \frac{S_k^{p+1}}{4^{p+1} D_k^p H_k},$$

the last inequality, because $l \leq k$ and both $D_k/S_k$ and $H_k/D_k$ are non-decreasing. Thereby,

$$t_{H_k/D_k}(\mathcal{Y}) \leq \frac{S_{\bar{Y}}}{S_k^{p+1}} 4^{p+1} D_k^p H_k,$$

that is $\bar{\mathcal{Y}}$ is sufficiently approximating of order $k$ and multiplicity $S_{\bar{Y}}$ with respect to $\theta$.

2. For $k = l$, since $\mathrm{div} f$ intersects $\mathcal{Y}$ properly, by the derivative metric Bézout Theorem (2.8),

$$\begin{aligned}
2D^{S_Y(S_k-1)/9^{p+1}}(\mathrm{div} f.Y,\theta) &\leq \max(S_k D^{S_Y/9^p}(Y,\theta), S_Y D^{(S_k-1)/9^p}(\mathrm{div} f,\theta)) \\
&+ 2H_k \deg Y + 2D_k h(\mathcal{Y}) + 2dD_k \deg Y \\
&+ c(D_k \deg Y)\log(D_k \deg Y) \\
\\
&\leq \max(S_k D^{S_Y/9^p}(Y,\theta), S_Y D^{(S_k-1)/9^p}(\mathrm{div} f,\theta)) \\
&+ 7D_k t_{H_k/D_k}(\mathcal{Y}) \log(D_k \deg Y).
\end{aligned}$$

Further, by (7), and Proposition 2.6,

$$S_k D^{S_Y/9^p}(Y,\theta) \leq -\frac{4S_Y V_k}{14^{p-1}},$$

$$S_Y D^{(S_k-1)/9^p}(\mathrm{div} f,\theta) \leq -5S_Y V_k + cD_k \log D_k \leq -4S_Y V_k,$$

and by (6) and (8), since $p \le t \le s-1$,

$$
\begin{aligned}
7D_k t_{H_k/D_k}(\mathcal{Y}) \log(D_k \deg Y) &\le 7 \cdot 4^p \frac{S_Y}{S_k^p} D_k^p H_k \log(D_k \deg Y) \\
&\le 7 \cdot 4^p S_Y V_K / C \log(D_k \deg Y) \le \frac{S_Y V_k}{14^{p-1}},
\end{aligned}
$$

for $C$ sufficiently big. Hence,

$$
2D^{S_Y(S_k-1)/9^{p+1}}(\mathrm{div} f.Y, \theta) \le -\frac{4S_Y V_k}{2 \cdot 14^{p-1}} = -\frac{4 \cdot S_Y S_k V_k}{2 \cdot 14^p S_k},
$$

that is

$$
\varphi^{S_Y(S_k-1)/9^{p+1}}_{\frac{H_k}{D_k}}(\mathrm{div} f.\mathcal{Y}, \theta) \le -\frac{4S_k S_Y V_k}{2 \cdot 14^p S_k t_{\frac{H_k}{D_k}}}(\mathrm{div} f.\mathcal{Y}).
$$

Thereby the premisses of part 1 are fulfilled with $l = k$, and part one implies the claim.
If $l < k$, and $|\mathrm{div} f, \theta| \le |Y, \theta|$, the claim follows similarly, this time using Corollary 2.9.2.

Proof of Theorem 1.2 Assume $t \le s+1$, let $k_0 \in \mathbb{N}$ be any number, and

$$
R = \inf\{\log |\mathrm{div} f, \theta| \; | f \in \Gamma(\mathbb{P}^t, O(D_{k_0})), \log |f| \le H_{k_0}, f \ne 0\}.
$$

Let further $C$ be an arbitrarily big constant, and $k > k_0$ such that

$$
\frac{S_k^s V_k}{D_k^s(D_k + H_k)} \ge C, \tag{11}
$$

and

$$
\frac{S_k^{t-1} V_k}{D_k^{t-1}(D_k + H_k)} \ge CR.
$$

Let $\mathcal{Y} \subset \mathcal{X}$ be a subvariety of maximal codimension that is sufficiently approximating of order $k$ and some multiplicity $S_Y$. Then $Y$ is contained in the support of $\mathrm{div} f$ for every $f \in \mathcal{F}_k$, since otherwise, by Lemma 5.2.2, there would be a subvariety $\bar{\mathcal{Y}}$ of codimension $p+1$ fulfilling the same conditions, thereby contradicting the maximality of the codimension of $\mathcal{Y}$. Since the derivated algebraic distance of the zero cycle is defined as 0, we have $p \le t$.
Let now

$$
l = \max\{\bar{k} \le k | \exists f \in \mathcal{F}_{\bar{k}-1} : Y \not\subset \mathit{supp}(\mathrm{div} f)\},
$$

Then, $Y$ is contained in the support of $\mathrm{div} f$ for every $f \in \mathcal{F}_l$, hence

$$
\log |Y, \theta| > -V_{l-1}/(S_{l-1}), \tag{12}
$$

and for every $f \in \mathcal{F}_l$, by [Ma1], Theorem 2.2.1 and (6), and (7),

$$\log|\mathrm{div} f, \theta| \le \log|Y, \theta| \le \varphi_{H_k/D_k}(Y, \theta) + c \le \varphi_{H_k/D_k}^{S_Y/9^p}(Y, \theta) \le$$

$$-\frac{4S_Y V_k}{14^{p-1} t_{H_k/D_k}(\mathcal{Y}) S_k} \le -\frac{4S_Y V_k S_k^{p-1}}{14^{p-1} S_k S_Y D_k^{p-1}} H_k \le -\frac{4V_k S_k^{t-1}}{14^{p-1} D_k^{-t}(D_k + H_k)} \frac{S_{k_0}^{t-p}}{D_{k_0}^{t-p}} < -R,$$

the last inequality holding if the constant $C$ is chosen sufficiently big. The inequalities $\log|dif, \theta| < -R$ for every $f \in \mathcal{F}_l$ imply $l > k_0$.
Let $D = [S_{l-1} S_Y V_k / (14^{p-1} V_{l-1})]$. If $\deg Y \le D/3$, then, again by [Ma1], Theorem 2.2.1,

$$\log|Y, \theta| \le \frac{-S_Y V_k}{3 \cdot 14^{p-1} D} \le -V_{l-1}/3S_{l-1}$$

in contradiction with (12). If $\deg Y \ge D$, let $g \in \mathcal{F}_{l-1}$ be such that $Y \not\subset \mathrm{supp}(\mathrm{div} g)$. If $|\mathrm{div} g, \theta| \le |Y, \theta|$, Lemma 5.2.1 would contradict the minimality of the dimension of $\mathcal{Y}$. Hence, $|Y, \theta| \le |\mathrm{div} g, \theta|$.
Using Corollary 2.9 for $Z_0 = Y, Z_1 = \mathrm{div} g, d_0 = D, S_0 = S_Y, S_1 = S_{l-1}$, one gets a $K \le DS_{l-1}$ such that

$$K \log|Y, \theta| + 2D^{S_Y(S_{l-1}-1)/9^{p+1}}(Y.\deg g, \theta) \le$$

$$\max(D\ D^{S_{l-1}-1}(\mathrm{div} g, \theta), S_{l-1} D^{S_Y}(Y, \theta) +$$

$$2H_{l-1} \deg Y + 2D_{l-1} h(\mathcal{Y}) + 2dD_{l-1} \deg Y.$$

Since $D^{S_{l-1}-1}(\mathrm{div} g, \theta) \le -5V_{l-1}, D^{S_Y}(Y, \theta) \le -4S_Y V_k / 14^{p-1} S_k$, and by assumption $H_{l-1}/S_{l-1} \le H_k/S_k$, and $D_{l-1}/S_{l-1} \le D_k/S_k$, the above is less or equal

$$\max(-5S_{l-1} S_Y V_k / (2 \cdot 14^{p-1}), -S_{l-1} S_Y V_k / (14^{p-1})) +$$

$$H_k \frac{S_{l-1}}{S_k} \deg Y + D_k \frac{S_{l-1}}{S_k} h(\mathcal{Y}) + dD_k \frac{S_{l-1}}{S_k} \deg Y.$$

Further, by (6)

$$\begin{aligned} 2D_k \frac{S_{l-1}}{S_k} h(\mathcal{Y}) \le 2D_k \frac{S_{l-1}}{S_k} t_{\frac{H_k}{D_k}}(\mathcal{Y}) &\le 2D_k \frac{S_{l-1}}{S_k} \frac{4^p S_Y D_k^{p-1}(D_k + H_k)}{S_k^p} \\ &= 2S_{l-1} S_Y \frac{4^p D_k^p (D_k + H_k)}{S_k^{p+1}} \le 2 \cdot 4^p S_{l-1} S_Y \frac{V_k}{C}. \end{aligned}$$

The last inequality because of $p \le t$. Similarly,

$$2H_k \frac{S_{l-1}}{S_k} \deg Y \le 2 \cdot 4^p S_{l-1} S_Y \frac{V_k}{C}, \quad dD_k \frac{S_{l-1}}{S_k} \deg Y \le 2 \cdot 4^p S_{l-1} S_Y \frac{V_k}{C}.$$

Hence,

$$K \log|\mathrm{div} g + Y, \theta| + 2D^{9S_Y S_{l-1}/9}(Y.\deg g, \theta) \le$$

$$-5S_{l-1}S_YV_k/(2\cdot 14^{p-1})+6S_{l-1}S_YV_k/C \leq -S_{l-1}S_YV_k/(2\cdot 14^{p-1}),$$

for $C$ sufficiently large.
Since $\mathcal{Y}$ was chosen of maximal codimension, Lemma 5.2.1 implies $D^{S_YS_{l-1}/9}(Y.\mathrm{div}g,\theta) \geq -S_{l-1}S_YV_k/(4\cdot 14^{p-1})$. Consequently,

$$K\log|\mathrm{div}g+Y,\theta| \leq -S_{l-1}S_YV_k/(4\cdot 14^{p-1}),$$

and thereby

$$\log|Y,\theta| \leq -S_{l-1}S_YV_k/(4K\cdot 14^{p-1}).$$

Since $K \leq S_{l-1}D$, this is less or equal

$$-S_YV_k/(4D14^{p-1}) \leq -V_{l-1}/(4S_{l-1}),$$

again contradicting (12). Since the assumtions $t-1 \leq s$ leads to a contradiction, we have $t-1 > s$.

# 6 Proof of second criterion

To prove Theorem 1.7, let $\theta$ be a point in projective space $\mathbb{P}^M$, assume its algebraic closure $\mathcal{X}$ over Spec $\mathbb{Z}$ has relative dimension $t$, and let $(D_k,S_k,H_k,V_k)$ be a quadrupel of series fulfilling the assumptions of the Theorem. Let further $F,G$ be the functions $F(k)=D_k/S_k, G(k)=H_k/D_k$. Since $F,G$ are of uniform polynomial growth, by Lemma 1.4, there is a $k_0$ such that for every $k \geq k_0$,

$$\frac{1}{2}D_{k+1}/S_{k+1} \leq D_k/S_k \leq D_{k+1}/S_{k+1}, \quad \frac{1}{2}H_{k+1}/S_{k+1} \leq H_k/S_k \leq H_{k+1}/S_{k+1}. \quad (13)$$

By Lemma 1.4, the function $H(D)=G\circ F^{-1}(D)$ is of uniform polynomial growth with $n_H \geq 0$. Multiplying $H(D)$ by a positive constant, if necessary, one can assure that $H(D) \geq aD$ with an arbitrary number $a \geq 1$. By Proposition 2.11, there are numbers $b_1, 1 > c_0 > 0, n_1 \in \mathbb{N}$ and an infinite subset $M \subset \mathbb{N}$ such that for each $D \in M$ there is an irreducible variety $\beta_{nD}$ of codimension $t$ in $\mathbb{P}^t$ and a locally complete intersection $\mathcal{Z}$ at $\alpha_{nD}$ of codimension $r \leq t-1$, such that

$$\deg\beta_{n_1D} \leq (n_1D)^t, \quad h(\beta_{n_1D}) \leq H(n_1D)(n_1D)^{t-1}, \quad D(\beta_{n_1D},\theta) \leq -b_1t_H(\beta_{n_1D})D,$$
$$t_{H/D}(\beta_{n_1D}) \geq c_0t_{H/D}(\mathcal{Z}_{min})D^{t-r}, \quad (14)$$

where $\mathcal{Z}_{min}$ is the irreducible component of $\mathcal{Z}$ with minimal $\frac{H}{D}$-size. Let $\pi_X \to \mathbb{P}^t$ be the projection from section 4, and $\alpha_D \subset \mathcal{X}$ an irreducible component of $\pi_X^*\alpha_D$, further $\mathcal{Y}$ an irreducible component of $\pi_X^*\mathcal{Z}_{min}$ containing $\alpha_D$. By (14), Proposition 4.5.1, and Proposition 4.3, there are constants $b, 1 > c > 0, n \in \mathbb{N}$ such that

$$\deg\alpha_{nD} \leq (nD)^t, \quad h(\alpha_{nD}) \leq H(nD)(nD)^{t-1}, \quad D(\alpha_{nD},\theta) \leq -bt_H(\alpha_{nD})D,$$

$$t_{H/D}(\alpha_{nD}) \geq c t_{H/D}(\mathcal{Y}) D^{t-r}. \tag{15}$$

With a big constant $c_3$ put

$$c_1 = \frac{c}{9M(h(\mathcal{X}) + c_3 \deg X)}.$$

Since $\lim_{k\to\infty} \frac{V_k S_k^s}{D_k^s(D_k+H_k)} = \infty$, there is a $k_1 \geq k_0$ such that

$$\frac{V_k S_k^s}{D_k^s(D_k + H_k)} > 40Mh(\mathcal{X} + c_3 \deg X)(d+1)(2n \max(1/c_1, (10+d)/b))^t.$$

for every $k \geq k_1$, where $d$ is the constant from Proposition 2.12. Since $M$ is infinite, (13) implies that there is a $D \in M$ and a $k \geq k_1$ such that

$$\left(\frac{\min(c_1, b/(10+d))}{2}\right) D \leq \frac{D_k}{S_k} < (\min(c_1, b/(10+d)))\, D. \tag{16}$$

Applying the function $H = G \circ F^{-1}$ to both sides, and using that it is eventually non-decreasing, gives

$$\left(\frac{\min(c_1, b/(10+d))}{2}\right) H \leq \frac{H_k}{S_k} < (\min(c_1, b/(10+d)))\, H, \tag{17}$$

with $H = H(D)$. Adding both inequalities implies

$$\left(\frac{\min(c_1, b/(10+d))}{2}\right)(H + D) \leq \frac{H_k + D_k}{S_k} < \min(c_1, b/(10+d))(H+D) \leq$$
$$2\min(c_1, b/(10+d))H. \tag{18}$$

For a given global section $h \in \Gamma(\mathbb{P}^M, O(1))$ with $h_\theta \neq 0$, identify an $f \in \mathcal{F}_k$ with $f/h^{D_k} \in \mathbb{Q}(X)$.

**6.1 Lemma** *There is an $f \in \mathcal{F}_k$ such that for some $I$ with $|I| \leq 2S_k/3$ the restriction of $\partial^I f$ to $\alpha_{nD}$ is nonzero.*

Proof Assume the opposite, and inductively construe a chain of subvarieties

$$\mathcal{Y}_1 \supset \cdots \supset \mathcal{Y}_{t-r} = \alpha_{nD},$$

such that

$$\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)} \leq c^{i-1} D^{i-1} t_{\frac{H}{D}}(\mathcal{Y}), \quad i = 1, \ldots t-r,$$
$$\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)} \leq c^{i-1} D^{i-1} t_{\frac{H}{D}}(\mathcal{Y}), \quad i = 2, \ldots t-r,$$

in the following way: Since $\alpha_{nD}$ is contained in $Y$, by fact 3.7, we have $v_{\alpha_{nD}}(Y) \geq 1$, thus can choose $\mathcal{Y}_1 = \mathcal{Y}$. Assume $\mathcal{Y}_i$ is given, and fulfills the above estimate. Since $\alpha_{nD}$ is contained in $\mathcal{Y}_i$, by [Ma1], Theorem 2.2.2,

$$\log|\mathcal{Y}_j, \theta| \leq \log|\alpha_{nD}, \theta| \leq \frac{D(\alpha_{nD}, \theta)}{t_{\frac{H}{D}}(\alpha_{nD})} + O(1) \leq -bD + O(1). \tag{19}$$

Thus, for $k$ sufficiently large, the assumption in the Theorem asserts that there is an $f_i \in \mathcal{F}_k$ and a multi index $I_i$ with $|I_i| \leq S_k/3$ such that the restriction of $\partial^{I_i} f_i$ to $\mathcal{Y}_i$ is nonzero, and by Theorem 3.3, there are polynomials $P, f_{I_i}$ with $P|_X \neq 0$, thereby $P(\theta) \neq 0$, and by (19) also $P(\alpha_{nD}) \neq 0$, fulfilling

$$\deg f_{I_i} \leq \deg f_i + (2S-1)(M-t)\deg X \leq 2M \deg X D_k,$$

$$\begin{aligned}
\log|f_{I_i}| &\leq \log|f_i| + \log\deg f_i \\
&+ (2S-1)(M-t)(h(\mathcal{X}) + c_4 \deg X + \log\deg X) + \log(2S!) \\
&\leq (2M(h(\mathcal{X}) + c_3 \deg X)(H_k + D_k) \leq 3M(h(\mathcal{X}) + c_3 \deg X)H_k,
\end{aligned}$$

and

$$\partial^{I_i} f = \frac{f_{I_i}}{P^{2|I_i|-1}},$$

and thereby

$$\partial^J f_{I_i}(\alpha_{nD}) = \partial^J(\partial^{I_i} f P^{2|I_i|-1})(\alpha_{nD}) = 0$$

for every $J$ with $|J| \leq S_k/3$. Hence, by Proposition 3.16 $v_{\alpha_{nD}}(\mathrm{div} f_{I_i}) \geq S_k/3$, and by the local Bézout Theorem,

$$v_{\alpha_{nD}}(Y_i.\mathrm{div} f_{I_i}) \geq \frac{S_k}{3} v_{\alpha_{nD}}(Y_i).$$

Further, by the algebraic Bézout Theorem,

$$\deg(Y_i.\mathrm{div} f_{I_i}) \leq 2M \deg X D_k \deg Y_i,$$

$$h(\mathcal{Y}_i.\mathrm{div} f_{I_i}) \leq$$

$$2M \deg X D_k h(\mathcal{Y}_i) + 3M(h(\mathcal{X}) + c_3 \deg X)H_k \deg Y_i + 2cM \deg X D_k \deg Y_i \leq$$

$$2M \deg X D_k h(\mathcal{Y}_i) + 4M(h(\mathcal{X}) + c_3 \deg X)H_k \deg Y_i.$$

Hence,

$$t_{\frac{H}{D}}(\mathcal{Y}_i.\mathrm{div} f_{I_i}) \leq$$

$$2\frac{H}{D} M \deg X D_k \deg Y_i + 2M \deg X D_k h(\mathcal{Y}_i) + 4M(h(\mathcal{X}) + c_3 \deg X)H_k \deg Y_i \leq$$

$$2M \deg X D_k t_{\frac{H}{D}}(\mathcal{Y}_i) + 4M(h(\mathcal{X}) + c_3 \deg X)\frac{H_k}{H} D\, t_{\frac{H}{D}}(\mathcal{Y}_i).$$

Together with the above estimate on the order of vanishing of $Y_i.\mathrm{div} f_{I_i}$ at $\alpha_{nD}$, this gives

$$\frac{t_{\frac{H}{D}}(\mathcal{Y}_i.\mathrm{div} f_{I_i})}{v_{\alpha_{nD}}(Y_i.\mathrm{div} f_{I_i})} \leq \frac{3}{S_k}\left(2M \deg X D_k + 4M(h(\mathcal{X}) + c_3 \deg X)\frac{H_k}{H} D\right)\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)},$$

which by (16), and (17) is less or equal

$$\left(\frac{\min(c_1, b)}{2}\right) D\left(2M \deg X + 4M(h(\mathcal{X}) + c_3 \deg X)\right)\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)} <$$

$$\left(\frac{\min(c_1, b)}{2}\right) D\left(3M \deg X + 5M(h(\mathcal{X}) + c_3 \deg X)\right)\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)} \leq$$

$$cD\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)}.$$

By Lemma 3.15, there is an irreducible component $\mathcal{Y}_{i+1}$ of $\mathcal{Y}_i \mathrm{div} f_{I_i}$ such that

$$\frac{t_{\frac{H}{D}}(\mathcal{Y}_{i+1})}{v_{\alpha_{nd}}(Y_{i+1})} \leq \frac{t_{\frac{H}{D}}(\mathcal{Y}_i.\mathrm{div} f_{I_i})}{v_{\alpha_{nd}}(Y_i.\mathrm{div} f_{I_i})},$$

which by the above is less than

$$cD\frac{t_{\frac{H}{D}}(\mathcal{Y}_i)}{v_{\alpha_{nD}}(Y_i)},$$

which by induction hypothesis is less or equal

$$c^i D^i t_{\frac{H}{D}}(\mathcal{Y}).$$

proving the claim for $i+1$. For $i = t - r$, the claim gives

$$t_{\frac{H}{D}}(\alpha_{nD}) = \frac{t_{\frac{H}{D}}(\alpha_{nD})}{v_{\alpha_{nD}}(\alpha_{nD})} < c^{t-r} D^{t-r} t_{\frac{H}{D}}(\mathcal{Y}),$$

contradicting the lower estimate on $t_{\frac{H}{D}}(\alpha_{nD})$ in (15)

Proof of Theorem 1.7, continuation Let $g = \partial^I f$ with $|I| \leq 2S_k/3$ be as in the Lemma. By Theorem 3.3, there are polynomials $P, g_I$ such that with $c_3$ chosen sufficiently big,

$$g = \frac{g_I}{P^{2|I|-1}}, \quad \deg g_I \leq 2M \deg X D_k,$$

$$\log|g_I| \leq 3M(h(\mathcal{X}) + c_3 \deg X)H_k,$$

and by Corollary 3.4,

$$\sup_{|J|\leq S_k/3} \log|\langle \partial^I g_I|\theta\rangle| \leq -V_k/2.$$

Theorem 3.3 also implies $P|_X \neq 0$, and thereby $P(\theta) \neq 0$, which by (19) implies

$$P(\alpha_{nD}) \neq 0.$$

Hence, by Theorem 2.8.2, and Corollary 3.4,

$$D(\alpha_{nD}, \mathrm{div} g_I) \leq \max\left(\frac{S_k}{3} D(\alpha_{nD}, \theta), -V_k/2\right) \leq \max\left(-b\frac{S_k}{3} D t_{H_D}(\alpha_{nD}), -V_k/2\right).$$

Further, by Liouvilles Theorem 2.12,

$$\begin{aligned}
D(\alpha_{nD}, \mathrm{div} g_I) \;&\geq\; -2M\deg X D_k h(\alpha_{nD}) - \deg\alpha_{nD} 3M(h(\mathcal{X}) + c_3\deg X)H_k \\
&\quad -2dM\deg X D_k \deg\alpha_{nD} \\
&\geq\; -2M\deg X D_k t_{\frac{H}{D}}(\alpha_{nD}) - 3M(h(\mathcal{X}) + c_3\deg X)\frac{H_k D}{H} t_{\frac{H}{D}}(\alpha_{nD}) \\
&\quad -2dM\deg X D_k \frac{D}{H} t_{\frac{H}{D}}(\alpha_{nD}) \\
&\geq\; -\left(2(d+1)M\deg X D_k + 3M(h(\mathcal{X}) + c_3\deg X)\frac{H_k D}{H}\right) \times \\
&\quad t_{\frac{H}{D}}(\alpha_{nD}).
\end{aligned}$$

The two inequalities together give

$$-\left(2(d+1)M\deg X D_k + 3M(h(\mathcal{X}) + c_3\deg X)\frac{H_k D}{H}\right) t_{\frac{H}{D}}(\alpha_{nD}) \leq$$

$$\max\left(-b\frac{S_k}{3} D t_{\frac{H}{D}}(\alpha_{nD}), -V_k/2\right).$$

If $-(2M(d+1)\deg X D_k + 3M(h(\mathcal{X}) + c_3\deg X)\frac{H_k D}{H}) t_{H_D}(\alpha_{nD})$ were less or equal $-b(S_k/3) D t_{\frac{H}{D}}(\alpha_{nD})$, if $c_3$ is chosen sufficiently big, this would contradict the second inequality of (18). Hence,

$$-\left(2(d+1)M\deg X D_k + 3M(h(\mathcal{X}) + c_3\deg X)\frac{H_k D}{H}\right) t_{H_D}(\alpha_{nD}) \leq -V_k/2. \quad (20)$$

By the upper estimates on $\deg\alpha_{nD}$, and $h(\alpha_{nD})$,

$$\left(2(d+1)M\deg X D_k + 3M(h(\mathcal{X}) + c_3\deg X)\frac{H_k D}{H}\right) t_{H_D}(\alpha_{nD}) \leq$$

$$2(2M\deg X(d+1)D_k+3M(h(\mathcal{X}+c_3\deg X)\frac{H_kD}{H})2Hn^tD^{t-1},$$

which by (16) and (17) is less or equal

$$8M\deg X(d+1)(2n)^t\max(1/c_1,(10+d)/b))^t\frac{H_kD_k^t}{S_k^t}+$$

$$12M(h(\mathcal{X}+c_3\deg X)(2n)^t\max(1/c_1,(10+d)/b))^t\frac{H_kD_k^t}{S_k^t}\leq$$

$$18Mh(\mathcal{X}+c_3\deg X)(d+1)(2n^t)\max(1/c_1,(10+d)/b))^t\frac{H_kD_k^t}{S_k^t},$$

for $c_3$ sufficiently big. Together with (20), this implies

$$\frac{V_kS_k^t}{D_k^tH_k}\leq 40Mh(\mathcal{X}+c_3\deg X)(d+1)(2n\max(1/c_1,(10+d)/b))^t.$$

Since $k$ was chosen such that

$$\frac{V_kS_k^s}{D_k^s(S_k+H_k)}>40M(h(\mathcal{X})+c_3\deg X)(d+1)(2n\max(1/c_1,(10+d)/b))^t,$$

and $S_k/D_k<1$, we get $t\geq s+1$.

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